FOLIATION CONES II

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Abstract. This paper substantially generalizes our paper "Foliation Cones" (KirbyFest, 1999) while simplifying many proofs and correcting some errors and gaps. In essence, we classify finite depth, foliated 3-manifolds $(M, \mathcal{F})$ with a given "substructure" S. The components W of $M \smallsetminus S$ are stably foliated and the possible such foliations are classified, up to isotopy, by the rays through the integer lattice points in the interiors of finitely many closed, convex, non-overlapping, finite-sided, polyhedral cones in a suitable cohomology of W. The other rays classify dense-leaved refoliations of W without holonomy, also up to isotopy, as will be shown in a paper "Foliation Cones III" now in preparation. While the cones are generally infinite dimensional, they have only finitely many faces. Results of this type are given both for the case that $(M, \mathcal{F})$ is smooth and that in which it is of class C^{0+} (i.e., $\mathcal{F}$ is integral to a C^0 plane field).

1. Introduction

We assume that M is a compact 3-manifold and that $\mathcal{F}$ is a transversely oriented, codimension 1, finite depth foliation of M. We will assume that $\mathcal{F}$ has depth $k \geq 1$, hence does not fiber M over S^1 or I. The manifold and hence the leaves may be nonorientable. If the foliated manifold $(M, \mathcal{F})$ is smooth of class C^2, then it is homeomorphic to a C^∞-foliated manifold [7, 10]. Indeed, implicit in these references is the fact that each leaf of the C^∞-foliated manifold can be assumed to have holonomy C^∞-tangent to the identity ι (such a foliation will be said to be of class $C^{\infty,\iota}$). Accordingly, all C^2 foliations hereafter will be assumed to be of class $C^{\infty,\iota}$. We will also consider the case of minimal regularity, the foliation only being tangent to a C^0 plane field, saying that such a foliation is of class C^{0+}. (In [3] such foliations were said to be of class $C^{1,0+}$ to emphasize that the leaves are individually of class C^1. In fact, for codimension 1 foliations, the leaves are individually of class C^∞ [10].)

Our foliated manifolds will be allowed to have tangential boundary $\partial_\tau M$, with components compact leaves, and/or transverse boundary $\partial_\pitchfork M$ with components annuli, tori, and/or Klein bottles. (These are sutured manifolds in the sense of D. Gabai [19].) The tangential and transverse boundary are separated by convex corners along their common boundary. The foliation

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induced by $\mathcal{F}$ on $\partial_\pitchfork M$ will be assumed to have no Reeb components. We will assume that no compact leaf is an annulus, Möbius strip, torus or Klein bottle. Because of Reeb stability, we rule out the totally trivial cases of a compact leaf being a disk, sphere, or projective plane. For proper, C^2 foliations, our hypotheses imply that no leaf has a simple end [4, Definition 2.1]. By [4, Lemma 2.19] this remains true in the C^{0+} case. In either case, our hypotheses imply that the foliation is taut [4, Lemma 2.18] and that there is a Riemannian metric on M such that all leaves of $\mathcal{F}$ are hyperbolic [2, 8].

Definition 1.1. A leaf of $\mathcal{F}$ is locally stable if it has a saturated neighborhood foliated as a product.

The union $\mathcal{O}$ of the locally stable leaves is open and dense. The complement $S = M \smallsetminus \mathcal{O}$ is called the *substructure* of $\mathcal{F}$. While $\mathcal{O}$ may have infinitely many components W, at most finitely many fail to be "foliated products". We say that W is a foliated product if its transverse completion $\widehat{W}$ has the structure of an interval bundle with $\mathcal{F}|\widehat{W}$ transverse to the fibers. In any event, $\mathcal{F}|W$ fibers W over a 1-manifold. The transverse completion of W is obtained by adding finitely many border leaves making up $\partial_\tau \widehat{W}$. (It is possible that pairs of distinct components of $\partial_\tau \widehat{W}$ are identified to the same leaf of the substructure SßM.) Remark that, for finite depth foliations, the leaves at maximal depth are locally stable. These facts are all standard, being found, for instance, in [3].

Definition 1.2. The subspace of $H^1(\widehat{W})$ (real coefficients) consisting of classes that can be represented by compactly supported cocycles will be denoted by $H^1_\kappa(\widehat{W})$.

Remark. This is not the same as the cohomology with compact supports, denoted by $H^1_c(\widehat{W})$. The natural homomorphism from this latter space to the former is surjective but not generally injective.

The subspaces $H^1(\widehat{W};\mathbb{Z})$ and $H^1_\kappa(\widehat{W};\mathbb{Z}) = H^1_\kappa(\widehat{W}) \cap H^1(\widehat{W};\mathbb{Z})$ of $H^1(\widehat{W})$ and $H^1_\kappa(\widehat{W})$, respectively, are called the integer lattices in these vector spaces.

Definition 1.3. The rays in $H^1(\widehat{W})$ and $H^1_\kappa(\widehat{W})$ issuing from the origin and meeting nonzero points of the integer lattice are called rational rays. Rays not meeting nonzero points of the integer lattice are termed irrational.

If $\widehat{W}$ is not compact, the spaces $H^1(\widehat{W})$ and $H^1_\kappa(\widehat{W})$ are infinite dimensional and will be given appropriate topologies.

Our principal goal is the proof of the following result.

Theorem 1.4. *Let $(M, \mathcal{F})$ be of class $C^{\infty,\iota}$ and depth $k \geq 1$. If W is a component of $\mathcal{O}$, there is a finite family $\mathcal{K}_W$ of nonoverlapping, closed, convex, finite-sided, polyhedral cones in $H^1_\kappa(\widehat{W})$, with nonempty interiors, having the following property: the set of rational rays in the interiors of*

these cones is in natural one-one correspondence with the set of isotopy classes of taut foliations $\mathcal{F}'$ of $\widehat{W}$, having holonomy only along $\partial_\tau \widehat{W}$, each of which completes $\mathcal{F}|(M \smallsetminus W)$ to a $C^{\infty,\iota}$ foliation of depth k. This isotopy is smooth in W but may only be C^0 on $\widehat{W}$. When $(M, \mathcal{F})$ is of class C^{0+}, the entirely analogous result holds, where the cones are contained in the ordinary cohomology $H^1(\widehat{W})$.

The elements of $\mathcal{K}_W$ are called *foliation cones*. We will denote the ray corresponding to $\mathcal{F}'$ by $\langle \mathcal{F}' \rangle$ and call it a foliated ray. If it is a rational ray, it will also be called a proper foliated ray, the adjective "proper" referring to the fact that the leaves of $\mathcal{F}'$ are proper. In the case that $\mathcal{F}$ is of class $C^{\infty,\iota}$, we will say that the foliation $\mathcal{F}'$ "appropriately extends" $\mathcal{F}|(M \smallsetminus W)$ if the new foliation obtained by replacing $\mathcal{F}|\widehat{W}$ with $\mathcal{F}'$ is also of class $C^{\infty,\iota}$.

Remark. In principle, this theorem allows us to classify homologically all finite depth foliations with a given substructure S. In the smooth case, it is known that the foliation in each component W of $\mathcal{O}$ must be trivial outside of some compact subset of $\widehat{W}$ (compactness of junctures [3, Theorem 8.1.26]), which is exactly why the cones live in $H^1_\kappa(\widehat{W})$ in that case.

Remark. It will be necessary to allow the possibility that the entire vector space $H^1_\kappa(\widehat{W})$ is a foliation cone, this happening if and only if $\widehat{W}$ is a foliated product. This is the one case in which the vertex 0 of the cone lies in its interior. The class 0 will correspond to the product foliation and $\{0\}$ will be a (degenerate) foliated ray.

Remark. The case in which the foliation is depth one with only the components of $\partial_\tau M$ as compact leaves is worth special mention. The problem of classifying them was proposed by W. Thurston in the closing paragraph of his Memoir paper [34]. This paper found polyhedral cones classifying fibrations transverse to ∂M (depth 0 foliations) subtended by the "fibered faces" of the Thurston ball. There was the clear expectation there that the classification in the depth one case would again use his norm on $H^1(M)$ or, perhaps, a similar norm. The solution to this classification problem, given in [11] and generalized here, does not use a norm. Indeed, examples such as the complement of the knot 8_{15} and the link $(2, 2, 2)$ show that, in general, the cones cannot be defined by a norm. There are indications that our cone structures for this case relate to the polytopes constructed by A. Juhász in his work on sutured Floer homology [26].

Remark. In a paper [12], now in preparation, we prove that the irrational rays in the interiors of the cones correspond one-to-one to the C^0 isotopy classes of foliations of $\widehat{W}$ which are dense-leaved without holonomy in W. More precisely, every such foliation is C^0 isotopic to one defined by a "foliated form" (Definition 3.13), and those defined by foliated forms are unique up to an isotopy that is C^∞ in W. Thus, we will obtain a generalization of

the Laudenbach-Blank theorem [27]. That theorem stated that, on a compact 3-manifold, nonsingular, cohomologous, closed 1-forms are smoothly isotopic, and we will prove this for the foliated forms on W.

Remark. From the point of view of 3-manifold topology, classification of the possible substructures would seem to be of considerable interest. Gabai's sutured manifold decompositions [20] appear basically to do that. It would be interesting if the methods of this paper would lead to a more elementary algorithm. These methods might also clarify the obstructions to achieving smoothness in the resulting foliations.

The proof of Theorem 1.4 uses the Handel-Miller classification of endperiodic homeomorphisms [4] and the Schwartzmann-Sullivan theory of asymptotic cycles [32, 33]. The results proven exclusively by asymptotic cycles are valid for compact manifolds of arbitrary dimension $n \geq 3$ with finite depth foliations. These results promise to be useful for further research. The Handel-Miller theory is strictly for 2-dimensional leaves and there is no evident replacement for it in higher dimensions. Although Handel and Miller never published the details, a complete account is now available [4].

In the case of depth 1 foliations arising from disk decompositions [19], if the disks can be chosen in M from the start, our classification program is quite effective. Indeed, the disks of the decomposition typically split up in a natural way into the rectangles of a Markov partition associated to the Handel-Miller monodromy and the foliation cones are easily determined from this information.

This paper can be read almost independently of its predecessor [11]. The one exception is that the proof of finiteness of the set of cones is so easily adapted to our present context that we will refer the reader to [11, Section 6] for details. The reader may also find it helpful to study the computations of foliation cones in [11, Section 7] and [5, Section 5] for a number of depth 1 examples.

2. Endperiodic monodromy

Let $\mathcal{F}$ be depth k of class C^2 or class C^{0+}. For the time being, we allow $\dim M = n \geq 3$. The definitions of the locally stable set $\mathcal{O}$ and the substructure S are valid in all dimensions. By [7, 10], we lose no generality in the C^2 case by assuming that $\mathcal{F}$ is of class $C^{\infty,\iota}$. Let $\mathcal{L}$ be a smooth 1-dimensional foliation of M that is transverse to $\mathcal{F}$ and is tangent to $\partial_\pitchfork M$, is oriented by the transverse orientation of $\mathcal{F}$, and induces the product foliation by compact intervals on all components of $\partial_\pitchfork M$ that have boundary and a foliation by circles on those components without boundary. (This, of course, restricts the topology of $\mathcal{F}$. In 3-manifolds it is guaranteed by our assumption that the induced foliation of $\partial_\pitchfork M$ has no Reeb components.) Let W be a component of $\mathcal{O}$.

There is a sublamination $\mathcal{XBL}|\widehat{W}$ consisting of those leaves that do not meet $\partial_\tau \widehat{W}$. This sublamination is a compact sublamination of $\mathcal{L}|W$. Indeed,

$\mathcal{X}$ is closed since every leaf of $\mathcal{L}|\widehat{W}$ that gets sufficiently close to $\partial_\tau \widehat{W}$ meets $\partial_\tau \widehat{W}$. The octopus decomposition of $\widehat{W}$ [3, Definition 5.2.13] implies that $\mathcal{X}$ is contained in a compact core K of $\widehat{W}$, hence is compact. We call $\mathcal{X}$ the *core lamination* of $\mathcal{L}|W$. We will assume that W is not a foliated product, hence the core lamination is nonempty. Indeed, if some leaf ℓ of $\mathcal{L}$ issues from one component of $\partial_\tau \widehat{W}$ but never reaches another component, the asymptote of ℓ in $\widehat{W}$ will be a nonempty subset of $\mathcal{X}$. Thus, if $\mathcal{X} = \emptyset$, every leaf of $\mathcal{L}$ issues from one component of $\partial_\tau \widehat{W}$ and terminates at another component, implying that $\widehat{W} \cong F \times I$, where F is a leaf of $\mathcal{F}$. In this case, we will see that Theorem 1.4 is rather trivial, hence we will assume that $\mathcal{X} \neq \emptyset$.

The foliation $\mathcal{L}|W$ can be parametrized as a flow Φ_t preserving $\mathcal{F}|W$ such that Φ_1 sends each leaf to itself. Indeed, $\mathcal{F}|W$ defines a fibration

$$\pi : W \to S^1$$

and one can lift the standard locally defined parameter θ of S^1 to $\mathcal{L}$. Equivalently, this is a smooth, transverse, holonomy invariant measure for $\mathcal{F}$. Remark that the parameter becomes unbounded near $\partial_\tau \widehat{W}$ and that $\mathcal{L} \pitchfork \partial_\tau \widehat{W}$, hence our flow extends to a flow on $\widehat{W}$ that fixes $\partial_\tau \widehat{W}$ pointwise. If L is a leaf of $\mathcal{F}|W$, the flow induces a first return map, $\Phi_1 = f : L \to L$, called the monodromy diffeomorphism of L. If $\mathcal{F}$ is of class C^2, the monodromy is endperiodic ([4, Definition 2.10 and Proposition 2.16] which extend to the higher dimensional cases). In the case that $\mathcal{F}$ is of class C^{0+}, however, the restriction $\mathcal{F}|K$ to the nucleus of an octopus decomposition has the property that the monodromy of any noncompact leaf is endperiodic and this will be sufficient for our purposes in that case.

Let $\mathcal{L}_f$ denote $\mathcal{L}|W$ and (by abuse) $\mathcal{L}|\widehat{W}$, recording thereby the monodromy f that the flow induces on L. We will also denote $\mathcal{X}$ß$\mathcal{L}$ by $\mathcal{X}_f$. If $g : L \to L$ is an endperiodic homeomorphism (*not* assumed to be smooth) isotopic to f through endperiodic homeomorphisms that agree with f on ∂L, we can again find a 1-dimensional foliation $\mathcal{L}_g$ of $\widehat{W}$, having the fibers of π as cross-sections, agreeing with $\mathcal{L}_f$ on $\partial_\pitchfork \widehat{W}$ and inducing monodromy $g : L \to L$. If g is not smooth, $\mathcal{L}_g$ cannot be smooth. We again get a core lamination $\mathcal{X}_g$ß$\mathcal{L}_g$, the saturation of the compact set X_gßL consisting of those points whose g-orbits are bounded away from all ends of L. If g is smooth (or, more generally, regular as defined below), we can apply the Schwartzmann-Sullivan theory to $\mathcal{X}_g$. Otherwise, only a very attenuated version of that theory applies.

3. The Schwartzmann-Sullivan asymptotic cycles

We continue to work in n-manifolds, $n \geq 3$, save mention to the contrary. For compact, 1-dimensional laminations such as $\mathcal{X}_g$ß$\mathcal{L}_g$, where g is smooth, Sullivan's theory of foliation cycles [33] specializes to the Schwartzmann theory of asymptotic cycles [32]. In fact, it is only required that $\mathcal{X}_g$ be integral to a C^0 vector field. If g is an endperiodic homeomorphism admitting $\mathcal{L}_g$

of class C^{0+}, we will say that g is *regular* and assume this save mention to the contrary. We will take as our main reference the exposition of Sullivan's theory in [3, Chapter 10], but taking into account the full theory of de Rham currents in [15, Chapitre III] on noncompact manifolds such as $\widehat{W}$. As developed in [3], this theory was concerned primarily with compact foliated manifolds having all leaves without boundary, but as noted there, everything goes through for compact laminations of possibly noncompact manifolds, provided that, again, all leaves have empty boundary.

3.1. **Forms and currents.** Slightly modifying the notation of [15] so as to keep track of the degrees of forms and currents, we set $\mathcal{D}_p = \mathcal{D}_p(\widehat{W})$, the locally convex topological vector space of *compactly supported* p-forms of class C^∞. That is, the underlying vector space is $A^p_c(\widehat{W})$ and the topology is generated by the increasing union of the topologies $\mathfrak{T}_k$ defined by the C^k norm $\|\cdot\|_k$ on compact sets, $0 \leq k < \infty$. This norm arises from a choice of C^∞ atlas on $\widehat{W}$, and basic open neighborhoods of 0 in $\mathfrak{T}_k$ are of the form $V(C,\varepsilon) = \{\varphi \mid \operatorname{supp}\varphi \subset C \text{ and } \|\varphi\|_k < \varepsilon\}$, where $C \subset \widehat{W}$ is compact and $\varepsilon > 0$. Following de Rham [15, p. 44], we say that a subset $B \subset \mathcal{D}_p$ is bounded if all of its elements have support in a common compact set and B is bounded in the C^k norm, $0 \leq k < \infty$.

We set $\mathcal{D}'_p$ equal to the strong dual of $\mathcal{D}_p$, the space of continuous linear functionals on $\mathcal{D}_p$. This is the space of *p-currents* on $\widehat{W}$. The topology on $\mathcal{D}'_p$ can now be defined exactly as in [3, Definition 10.1.17]. This makes $\mathcal{D}'_p$ into a locally convex, topological vector space. It has a notion of bounded subset [3, Definition 10.1.20]. Both $\mathcal{D}_p$ and $\mathcal{D}'_p$ are strong duals of one another [15, p. 89, Théorème 13].

De Rham also introduces a locally convex topological vector space $\mathcal{E}_p$ with underlying vector space $A^p(\widehat{W})$. The topology uses a notion of boundedness which is local boundedness in the C^k norms, enabling one to define when a sequence of forms converges to 0. The support of a p-current T is the smallest closed subset $C \subset \widehat{W}$ such that $T(\alpha) = 0$ whenever $\operatorname{supp}(\alpha) \cap C = \emptyset$. The subset $\mathcal{E}'_p \subset \mathcal{D}'_p$ of currents with compact support is exactly the strong dual of $\mathcal{E}_p$ and vice-versa [15, p. 89, Théorème 13]. Using the notion of bounded set in $\mathcal{E}_p$, one mimics [3, Definition 10.1.17] to define the locally convex topology on $\mathcal{E}'_p$.

All of these spaces are Montel, meaning that every bounded subset is precompact. For the case $p = 0$, this is proven in [31, p. 70, Théorèm VII and p. 74, Théorème XII], the general case being similar.

The exterior derivative $d : \mathcal{D}_p \to \mathcal{D}_{p+1}$ is continuous, hence has continuous adjoint $\partial : \mathcal{D}'_p \to \mathcal{D}'_{p-1}$. Similarly, one has that $d : \mathcal{E}_p \to \mathcal{E}_{p+1}$ is continuous with continuous adjoint $\partial : \mathcal{E}'_p \to \mathcal{E}'_{p-1}$. Since $d^2 = 0$, we see that $\partial^2 = 0$. The kernel $\mathcal{Z}_p \subset \mathcal{E}'_p$ of ∂ is the space of p-cycles and the image $\mathcal{B}_p = \partial(\mathcal{E}'_{p+1}) \subset \mathcal{E}'_p$ is the space of p-boundaries. These are closed subspaces of $\mathcal{E}'_p$. The space

$H_p(\widehat{W}) = \mathcal{Z}_p/\mathcal{B}_p$ is the de Rham homology of $\widehat{W}$, canonically isomorphic to the singular homology. The homology of the complex $(\mathcal{D}'_*, \partial)$ gives the dual space to $H^p_c(\widehat{W})$, for each $p \geq 0$, which we may think of as homology computed with locally finite, but possibly infinite, chains and denote it by $H^\infty_p(\widehat{W}) = \mathcal{Z}^\infty_p/\mathcal{B}^\infty_p$, where, of course, $\mathcal{Z}^\infty_p$ is the kernel of ∂ in $\mathcal{D}'_p$ and $\mathcal{B}^\infty_p$ its image.

Since the inclusion $W \hookrightarrow \widehat{W}$ is a homotopy equivalence, we can identify the homology and cohomology of W with that of $\widehat{W}$.

3.2. **Topologies on homologies and cohomologies.** In homology, we topologize $H_p(\widehat{W}) = \mathcal{Z}_p/\mathcal{B}_p$ and $H^\infty_p(\widehat{W}) = \mathcal{Z}^\infty_p/\mathcal{B}^\infty_p$ with the quotient topology.

The manifold $\widehat{W}$, if noncompact, is an increasing union

$$K_0 \text{ß} K_1 \text{ß} \cdots \text{ß} K_i \text{ß} \cdots = \widehat{W},$$

where each K_i is a compact, sutured manifold which is the nucleus of an octopus decomposition of $\widehat{W}$. This can be used to define natural topologies on $H^p(\widehat{W})$ and $H^p_\kappa(\widehat{W})$.

We topologize $H^p(\widehat{W}) = \varprojlim H^p(K_i)$ [25, Proposition 3F.5] with the inverse limit topology. This is the topology induced from the Tychonov topology by the inclusion

$$\varprojlim H^p(K_i) \text{ß} H^p(K_0) \times H^p(K_1) \times \cdots \times H^p(K_i) \times \cdots$$

In fact, this is also the quotient topology induced by the complex $(\mathcal{E}_*, d)$ (exercise). In any event, it is clear that the elements of $H^p(\widehat{W})$, viewed as linear functionals on $H_p(\widehat{W})$, are continuous and so, this being the vector space dual, it is also the strong dual of $H_p(\widehat{W})$. Similarly, the elements of $H_p(\widehat{W})$, viewed as linear functionals on $H^p(\widehat{W})$, are continuous (exercise), so homology and cohomology are strong duals of each other.

Set $K_i^{\text{ø}}$ equal to the relative interior of K_i in $\widehat{W}$, noting that the inclusions $H^p_\kappa(K_i^{\text{ø}}) \hookrightarrow H^p_\kappa(K_{i+1}^{\text{ø}})$ induce a natural identification $H^p_\kappa(\widehat{W}) = \varinjlim H^p_\kappa(K_i^{\text{ø}})$. We give this space the direct limit (weak) topology. Equivalently, the topology on $H^p_\kappa(\widehat{W}) \text{ß} H^p(\widehat{W})$ is the induced topology (exercise).

3.3. **The asymptotic currents for $\mathcal{X}_g$.** The Dirac currents for $\mathcal{X}_g$ are the positively oriented, nontrivial tangent vectors to $\mathcal{X}_g$. These currents clearly have compact support. The closure in $\mathcal{D}'_1$ of the union of all positive linear combinations of Dirac currents is a closed, convex cone $\mathcal{C}_g \text{ß} \mathcal{D}'_1$, called the cone of *asymptotic currents*. This cone lies on one side of a hyperplane $H = \omega^{-1}(0)$, where $\omega : \mathcal{D}'_1 \to \mathbb{R}$ is a compactly supported 1-form integrable on a neighborhood U of $\mathcal{X}_g$ and defining $\mathcal{F}|U$. (The proof in [3, Lemma 10.2.3] goes through practically unchanged, just noting that the compactness of $\mathcal{X}_g$ replaces that of the ambient manifold.) Since $\mathcal{X}_g$ is compact, the asymptotic currents are compactly supported and we can view $\mathcal{C}_g \text{ß} \mathcal{E}'_1$. When working in

$\mathcal{D}'_1$, the continuous linear functionals are the compactly supported 1-forms, but when working in $\mathcal{E}'_1$, they are all 1-forms.

The base $\widehat{\mathcal{C}}_g = \mathcal{C}_g \cap \omega^{-1}(1)$ of the cone $\mathcal{C}_g$ is compact (both in $\mathcal{E}'_1$ and in $\mathcal{D}'_1$) by [3, Lemma 10.2.3]. Once again the proof goes through by the compactness of $\mathcal{X}_g$ and the fact that our spaces are Montel. Those continuous linear functionals $\eta : \mathcal{D}'_1 \to \mathbb{R}$ which are strictly positive on $\widehat{\mathcal{C}}_g$ are exactly the smooth, compactly supported 1-forms on $\widehat{W}$ which are transverse to $\mathcal{X}_g$ (meaning that they take a positive value on each Dirac current). Similarly, the continuous linear functionals $\eta : \mathcal{E}'_1 \to \mathbb{R}$, strictly positive on $\widehat{\mathcal{C}}_g$, are the smooth 1-forms on $\widehat{W}$ which are transverse to $\mathcal{X}_g$. Sullivan applies the Hahn-Banach theorem, using compactness of the base, to produce interesting 1-forms that are transverse to $\mathcal{X}_g$ (see [3, Subsection 10.2]).

3.4. **Cones defined by the asymptotic cycles.** The cone $\mathcal{C}_g \cap \mathcal{Z}_1$ of *asymptotic cycles* is also a closed, convex cone with compact base. The natural continuous linear surjection $\mathcal{Z}_1 \to H_1(\widehat{W})$ carries the cone of asymptotic cycles onto a convex cone $\mathfrak{C}'_g$ß$H_1(\widehat{W})$ with compact base. Compactness of the base implies that this cone is closed. There are dual closed, convex cones in $H^1_\kappa(\widehat{W})$ and $H^1(\widehat{W})$:

$$\mathfrak{C}^\kappa_g = \{[\eta] \in H^1_\kappa(\widehat{W}) \mid [\eta]([z]) \geq 0, \forall [z] \in \mathfrak{C}'_g)\},$$
$$\mathfrak{C}_g = \{[\eta] \in H^1(\widehat{W}) \mid [\eta]([z]) \geq 0, \forall [z] \in \mathfrak{C}'_g)\}.$$

Generally, these do not have compact base. Indeed, there are interesting cases in which $\mathfrak{C}'_g$ reduces to a single ray, hence the dual cone will be a full half-space.

Examples of asymptotic cycles are nonnegative, transverse, holonomy invariant measures μ on $\mathcal{X}_g$ that are finite on (transverse) compact sets. By Sullivan [3, Theorem 10.2.12], these are the only ones.

Theorem 3.1. *The asymptotic cycles for $\mathcal{X}_g$ are exactly the nonnegative, transverse, holonomy invariant measures on $\mathcal{X}_g$ that are finite on compact sets.*

By a well known theorem of J. F. Plante [29] and the fact that the leaves of $\mathcal{X}_g$, being 1-dimensional, have linear growth, we obtain the following.

Lemma 3.2. *There are nontrivial asymptotic cycles for $\mathcal{X}_g$.*

Lemma 3.3. *There is a closed 1-form on W, transverse to $\mathcal{X}_g$, hence no nontrivial asymptotic cycle bounds in $(\mathcal{E}'_*, \partial)$. If $\mathcal{F}$ is of class C^2, this form can be chosen to be compactly supported and no nontrivial asymptotic cycle bounds either in $(\mathcal{D}'_*, \partial)$ or $(\mathcal{E}'_*, \partial)$.*

Proof. Whether the foliation is of class C^2 or C^{0+}, $\mathcal{F}|W$, as a fiber bundle over S^1 with smooth leaves, has a smooth structure. Let ω be a closed, nonsingular 1-form defining $\mathcal{F}|W$. This is clearly transverse to $\mathcal{X}_g$ and ω takes positive values on all nontrivial asymptotic cycles which, therefore,

cannot bound in $(\mathcal{E}'_*, \partial)$. Relative to a suitable octopus decomposition of $\widehat{W}$, $\mathcal{X}_g$ is contained in the interior of the compact nucleus K_0 of $\widehat{W}$. If $\mathcal{F}$ is of class C^2, then $\mathcal{F}$ is trivial in $\widehat{W} \smallsetminus K_i$, for $i \geq 0$ sufficiently large (the C^2 hypothesis is critical here). Thus $\omega = d\gamma$ outside of K_i. Extending and damping γ off smoothly to be 0 in a neighborhood of $\mathcal{X}_g$, we see that the compactly supported form $\omega - d\gamma$ is as desired. Again ω takes positive values on all nontrivial asymptotic cycles, hence these cannot bound either in $(\mathcal{D}'_*, \partial)$ or $(\mathcal{E}'_*, \partial)$. □

We will need the following key result, the proof of which is an adaptation of the proof of [3, Lemma 10.2.8], which assumed the foliated manifold was compact. The other hypotheses of that lemma are satisfied because of Lemma 3.2 and Lemma 3.3.

Theorem 3.4. *Every nontrivial asymptotic cycle defines a nontrivial class in $\mathfrak{C}'_g$. Furthermore, $\operatorname{int}\mathfrak{C}_g \neq \emptyset$ and consists of exactly those classes $[\eta] \in H^1(\widehat{W})$ that are represented by closed 1-forms η transverse to $\mathcal{X}_g$. Furthermore, if $\mathcal{F}$ is of class C^2, the completely analogous assertion holds for $\mathfrak{C}^{\kappa}_g$.*

Proof. The nontriviality is given by Lemma 3.3. Let $\mathfrak{C}^{\emptyset}_g$ß$\mathfrak{C}_g$ be the set of classes that take strictly positive values on the nonzero elements of $\mathfrak{C}'_g$. This is nonempty by Lemma 3.3. If

$$[\eta] = ([\eta_0], [\eta_1], \dots, [\eta_i], \dots) \in \varprojlim H^1(K_i) = H^1(\widehat{W})$$

is in $\mathfrak{C}^{\emptyset}_g$, then η_0 takes positive values on all nontrivial asymptotic cycles. These cycles are all supported in the compact manifold K_0, hence the analogous result for $\mathcal{F}|K_0$ is proven in [3, Lemma 10.2.8]. (Compactness is used to guarantee that $H^1(K_0)$ is finite dimensional.) Thus $[\eta_0] \in U$ß$H^1(K_0)$, where U is the interior of the dual cone there. Then

$$\mathcal{U} = U \times H^1(K_1) \times H^1(K_2) \times \cdots \times H^1(K_i) \times \cdots$$

is an open set in the Tychonov topology and $\mathcal{U} \cap H^1(\widehat{W}) = \mathfrak{C}^{\emptyset}_g$, hence this set is open. We prove that it is the entire interior of $\mathfrak{C}_g$ by showing that every $[\alpha] \in \mathfrak{C}_g$ that vanishes on some nontrivial asymptotic cycle T is in the frontier. Indeed, if $[\eta] \in \mathfrak{C}^{\emptyset}_g$, form the line of classes $t[\eta] + (1-t)[\alpha]$, $-1 \leq t \leq 1$. For $0 < t \leq 1$, these classes are in $\mathfrak{C}^{\emptyset}_g$, but for $-1 \leq t < 0$, they take negative values on T, hence are not in $\mathfrak{C}_g$.

It remains to show that if $[\eta] \in \mathfrak{C}^{\emptyset}_g$, then η is cohomologous to a closed form η' that takes positive values on the entire base $\widehat{\mathcal{C}}_g$ of the cone of asymptotic currents. The kernel of $[\eta] : H_1(\widehat{W}) \to \mathbb{R}$ is a closed hyperplane in that vector space and its pre-image Vß$\mathcal{Z}_1$ is the kernel of $\eta : \mathcal{Z}_1 \to \mathbb{R}$, again a closed vector subspace containing $\mathcal{B}_1$ and meeting $\mathcal{C}_g \cap \mathcal{Z}_1$ only at the vertex of that cone, hence meeting $\mathcal{C}_g$ itself only at the vertex. Using the standard Hahn-Banach argument (cf. [3, Section 10.2]), we find a continuous linear functional $\eta' : \mathcal{E}' \to \mathbb{R}$ which is strictly positive on $\widehat{\mathcal{C}}_g$ and vanishes

identically on V. Since $\mathcal{B}_1$ßV, η' is a closed form transverse to $\mathcal{X}_g$. Since $[\eta']$ vanishes on the kernel of $[\eta]$ and is not trivial, $[\eta]$ and $[\eta']$ are positive multiples of each other, completing the proof for the case where $\mathcal{F}$ may only be of class C^{0+}.

If $\mathcal{F}$ is of class C^2, the proof is quite similar, but there are sufficient differences that we will give details. The fact that $\mathfrak{C}_g^{\kappa ø} \neq \emptyset$ is guaranteed by Lemma 3.3. The fact that this is the interior of $\mathfrak{C}_g^{\kappa}$ follows from the corresponding fact for $\mathfrak{C}_g$ via the relative topology. If $[\eta] \in \operatorname{int} \mathfrak{C}_g^{\kappa}$, we take η compactly supported, hence it is a continuous linear functional $\eta : \mathcal{D}_1' \to \mathbb{R}$. Restricting to the space $\mathcal{Z}_1^{\infty}$ of closed currents, we obtain a closed subspace $V = \eta^{-1}\{0\}$ which contains the space $\mathcal{B}_1^{\infty}$ of boundaries. Viewing $\mathcal{C}_g$ as a cone in $\mathcal{D}_1'$, we again apply the Hahn-Banach theorem to produce a compactly supported form η' strictly positive on $\widehat{\mathcal{C}}_g$ and vanishing on V. Thus η' is compactly supported and transverse to $\mathcal{X}_g$. Since $\mathcal{B}_1^{\infty}$ßV, η' is closed. Via the continuous injection $\mathcal{E}_1' \hookrightarrow \mathcal{D}_1'$, V pulls back to a space having as image in $H_1(\widehat{W})$ the kernel of $[\eta] : H_1(\widehat{W}) \to \mathbb{R}$. Since $[\eta']$ also vanishes on this space and is nontrivial, $[\eta]$ and $[\eta']$ are positive multiples of each other. $\square$

Varying the regular monodromy g within its isotopy class will give different cones $\mathfrak{C}_g$ and $\mathfrak{C}_g^{\kappa}$. One of our main goals is to show that, when $\dim M = 3$, there is a maximal one and that, in the smooth case, it corresponds to the Handel-Miller monodromy h in this isotopy class. Roughly speaking, this means that the Handel-Miller monodromy has the "tightest" dynamics in its isotopy class, determining the "narrowest" homology cone $\mathfrak{C}_h'$.

The notations $\mathfrak{C}_g$, $\mathfrak{C}_g^{\kappa}$ and $\mathfrak{C}_g'$, suggest dependence only on g, rather than on the choice of $\mathcal{X}_g$. In Subsection 3.6, we will see that this is correct when $\dim M = 3$. At the level of currents, the notation $\mathcal{C}_g$ is a bit of an abuse.

3.5. **Homology directions.** It will be important to characterize a particularly simple spanning set of $\mathfrak{C}_g'$, the so called "homology directions" of Fried [18, p. 260]. Assuming that $\mathcal{L}_g$ has been parametrized as a nonsingular C^{0+} flow Φ_t that preserves $\mathcal{F}|W$, select a point $x \in \mathcal{X}_g$ and let Γ denote the Φ-orbit of x. If this is a closed orbit, it defines an asymptotic cycle which we will denote by $\overline{\Gamma}$. If it is not a closed orbit, let $\Gamma_\tau = \{\Phi_t(x) \mid 0 \le t \le \tau\}$. Let $\tau_k \uparrow \infty$ and set $\Gamma_k = \Gamma_{\tau_k}$. After passing to a subsequence, we obtain an asymptotic current

$$\overline{\Gamma} = \lim_{k\to\infty} \frac{1}{\tau_k} \int_{\Gamma_k} .$$

In fact this is a *cycle*. One calls Γ_k a "long, almost closed orbit" of $\mathcal{X}_h$. Its endpoints lie in the compact set $\mathcal{X}_g$ and it can be closed by adding a uniformly bounded curve in M. These are averaged out in the limit and the corresponding singular cycles, also called "long, almost closed orbits", are denoted by Γ_k'. The cycles $(1/\tau_k)\Gamma_k'$ also limit on $\overline{\Gamma}$, proving that it is a cycle.

Lemma 3.5. *The asymptotic current* $\overline{\Gamma}$*, obtained as above, is an asymptotic cycle.*

Another proof can be given by appealing to Stokes's theorem.

Definition 3.6. All asymptotic cycles $\overline{\Gamma}$, obtained as above, and their homology classes are called homology directions of $\mathcal{X}_g$.

By abuse, we will denote both the cycle and its homology class by $\overline{\Gamma}$.

An elementary application of ergodic theory proves the following (see [33, Proposition II.25] and [3, Proposition 10.3.11]).

Lemma 3.7. *Any asymptotic cycle* μ *can be arbitrarily well approximated by finite, nonnegative linear combinations* $\sum_{i=1}^{r} a_i\overline{\Gamma}_i$ *of homology directions. If* $\mu \neq 0$*, the coefficients* a_i *are strictly positive and their sum is bounded below by a constant* $b_\mu > 0$ *depending only on* μ*.*

3.6. **The independence of the cones from various choices.** Let us first note that long, almost closed orbits and homology directions, as classes in the singular homology $H_1(\widehat{W})$, are clearly well defined even when $\mathcal{L}_g$ and $\mathcal{X}_g$ are only C^0 (which may well be the case when g is only a homeomorphism). While most of the theory of asymptotic classes fails when these objects are not at least integral to a C^0 line field, we can still define the cone $\mathfrak{C}'_g$ as the closure in $H_1(\widehat{W})$ of the set of nonnegative linear combinations of homology directions. This is a closed, convex cone which, when g is regular, coincides with the cone already defined (Lemma 3.7). We are going to show that, if $\dim M = 3$, this cone depends only on g, not on the choice of $\mathcal{L}_g$. In this subsection we require no regularity.

Let L be a leaf of $\mathcal{F}|W$, and let $\mathcal{L}$ and $\mathcal{L}_\sharp$ be 1-dimensional foliations of $\widehat{W}$ of class C^0, transverse to $\mathcal{F}|\widehat{W}$ and having each leaf of $\mathcal{F}|W$ as a section. Let $\mathcal{X}$ and $\mathcal{X}_\sharp$ be the respective core laminations and let $\mathfrak{C}'_{\mathcal{X}}$ and $\mathfrak{C}'_{\mathcal{X}_\sharp}$ denote the corresponding cones in $H_1(\widehat{W})$, $\mathfrak{C}_{\mathcal{X}}$ and $\mathfrak{C}_{\mathcal{X}_\sharp}$ those in $H^1(\widehat{W})$, and $\mathfrak{C}^\kappa_{\mathcal{X}}$ and $\mathfrak{C}^\kappa_{\mathcal{X}_\sharp}$ those in $H^1_\kappa(\widehat{W})$.

In [11], the following elementary theorem was deduced as a corollary of a much deeper result (Lemma 4.10 in that reference) which we attempted to deduce from results of M. E. Hamstrom [21, 22, 23]. A correct proof of that lemma needs a deep result of T. Yagasaki [36], but we omit this because we do not need it.

Theorem 3.8. *Let* L *be a leaf of* $\mathcal{F}|W$*, and let* $\mathcal{L}$ *and* $\mathcal{L}_\sharp$ *be* 1*-dimensional foliations of* $\widehat{W}$*, transverse to* $\mathcal{F}|\widehat{W}$ *and having each leaf of* $\mathcal{F}|W$ *as a section, having respective core laminations* $\mathcal{X}$ *and* $\mathcal{X}_\sharp$*, and inducing the same endperiodic monodromy* $g : L \to L$*. Then* $\mathfrak{C}'_{\mathcal{X}} = \mathfrak{C}'_{\mathcal{X}_\sharp}$*,* $\mathfrak{C}_{\mathcal{X}} = \mathfrak{C}_{\mathcal{X}_\sharp}$ *and* $\mathfrak{C}^\kappa_{\mathcal{X}} = \mathfrak{C}^\kappa_{\mathcal{X}_\sharp}$*.*

We will show that the homology directions determined by the long, almost closed orbits of $\mathcal{X}$ are exactly the same as the ones for $\mathcal{X}_\sharp$ and the theorem will follow. Our arguments are carried out in the C^0 category for arbitrary

surfaces L *which have nonabelian fundamental group.* Our leaf L, being noncompact with no simple end satisfies this requirement.

Let I be the compact interval $[0, 1]$ and consider the product $L \times I$. (One obtains such a product, for instance, by cutting W apart along L and taking as the interval fibers the resulting segments of the leaves of $\mathcal{L}$.) For each $x \in L$, denote by I_x the interval fiber with endpoints $\{x\} \times \{0, 1\}$. Consider a second fibration of $L \times I$ by intervals J_x, requiring that the endpoints of J_x coincide with those of I_x, for all $x \in L$. (By the hypothesis on $\mathcal{L}_\sharp$, this second fibration arises in our case by cutting apart along L and using the segments of leaves of $\mathcal{L}_\sharp$ as fibers.) For each $x \in L$, let α_x denote the loop in $L \times I$ obtained by following I_x from $(x, 0)$ to $(x, 1)$ and then following J_x from $(x, 1)$ to $(x, 0)$. Finally, if $p : L \times I \to L$ is the canonical projection, let $\beta_x = p \o \alpha_x$, a loop in L.

Lemma 3.9. *Let $x_0 \in L$ and set $\delta = \beta_{x_0}$. If $\gamma(s)$, $0 \le s \le 1$, is any other closed curve in L based at x_0, then $\gamma \cdot \delta = \delta \cdot \gamma$ in $\pi_1(L, x_0)$.*

Proof. Define $F(s, t) = \beta_{\gamma(s)}(t)$. Then $F(s, 0) = \gamma(s) = F(s, 1)$. Also $F(0, t) = F(1, t) = \beta_{x_0}(t) = \delta(t)$. Because of this last, we can view F as a map from the cylinder $S^1 \times [0, 1]$ into L. The curve obtained by following $F(0, t)$, $0 \le t \le 1$, followed by $F(s, 1)$, $0 \le s \le 1$, and then $F(0, 1 - t)$, $0 \le t \le 1$, is the composite loop $\delta \cdot \gamma \cdot \delta^{-1}$. We show how to deform this curve continuously to γ, keeping the basepoint x_0 fixed throughout the deformation.

Let σ_t be the curve obtained by following $F(0, \tau)$, $0 \le \tau \le t$, followed by $F(s, t)$, $0 \le s \le 1$, followed by $F(0, t - \tau)$, $0 \le \tau \le t$. Since $\sigma_0 = \gamma$ and $\sigma_1 = \delta \cdot \gamma \cdot \delta^{-1}$, we have the desired deformation. □

Corollary 3.10. *If there exists an $x_0 \in L$ so that J_{x_0} cannot be deformed into I_{x_0} keeping the endpoints fixed, then L is either the torus or has the homotopy type of the circle.*

Proof. The hypothesis implies that α_{x_0} is essential in $L \times I$, and so β_{x_0} is essential in L and thus is a nontrivial element of $\pi_1(L, x_0)$. By Lemma 3.9, every element of $\pi_1(L, x_0)$ commutes with β_{x_0}. The only closed orientable surface that L could be is the torus. If L is not closed, L is homotopically equivalent to a bouquet B of circles. The only bouquet of circles that contains a nontrivial element of $\pi_1(B, *)$ that commutes with every other element of $\pi_1(B, *)$ is one circle. □

Proof of Theorem 3.8. Let $\overline{\Gamma} \in H_1(\widehat{W}; \mathbb{R})$ be a homology direction for $\mathcal{X}$.

First assume that $\overline{\Gamma}$ is not represented by a closed orbit in $\mathcal{X}$ and write

$$\overline{\Gamma} = \lim_{k \to \infty} \frac{1}{\tau_k} [\Gamma'_k],$$

a limit in $H_1(M)$ of the homology classes of long, almost closed orbits. The numbers τ_k are the "lengths" of Γ_k (measured by the transverse, invariant measure for $\mathcal{F}|W$) and increase to ∞ with k. Thus, except for a uniformly

bounded arc in L, Γ'_k is a sequence of segments, $\sigma_1, \dots, \sigma_{n_k}$ of an orbit in $\mathcal{X}$, each starting and ending in L. There is a corresponding sequence $\sigma'_1, \dots, \sigma'_{n_k}$ of segments of an orbit in $\mathcal{X}_\sharp$ such that σ_i and σ'_i have the same endpoints and the same lengths, $1 \le i \le n_k$. By Corollary 3.10, these respective segments are homotopic by a homotopy that keeps their endpoints fixed. Thus, we see that $\overline{\Gamma}$ is also a homology direction for $\mathcal{X}_\sharp$. In the case that $\overline{\Gamma}$ is represented by a closed orbit, the argument adapts and is simpler. Finally, the roles of $\mathcal{X}$ and $\mathcal{X}_\sharp$ can be interchanged, proving that the two laminations have the same homology directions. □

Proposition 3.11. *Let $\mathcal{L}$ and $\mathcal{L}_\sharp$ be two 1-dimensional foliations transverse to $\mathcal{F}|\widehat{W}$. Suppose that the respective core laminations $\mathcal{X}$ and $\mathcal{X}_\sharp$ are C^0-isotopic by an isotopy $\varphi_t : \mathcal{X} \hookrightarrow M$, $\varphi_0 = \mathrm{id}_\mathcal{X}$ and $\varphi_1(\mathcal{X}) = \mathcal{X}_\sharp$, such that $\varphi_t(x)$ lies in the same leaf of $\mathcal{F}|W$ as x, for $0 \le t \le 1$, $\forall\, x \in \mathcal{X}$. Then $\mathfrak{C}'_\mathcal{X} = \mathfrak{C}'_{\mathcal{X}_\sharp}$ and $\mathfrak{C}_\mathcal{X} = \mathfrak{C}_{\mathcal{X}_\sharp}$.*

Proof. Parametrize the two foliations as flows using the same transverse invariant measure θ for $\mathcal{F}$. Since $\mathcal{F}$ is leafwise invariant under the isotopy, the flow parameter is preserved and the long, almost closed orbits of $\mathcal{X}$ are isotoped to the long, almost closed orbits of $\mathcal{X}_\sharp$. Homotopic singular cycles are homologous and the assertions follow. □

The property that points of W remain in the same leaf of $\mathcal{F}|W$ throughout the isotopy will be indicated by saying that $\mathcal{F}|W$ is leafwise invariant by φ_t.

Corollary 3.12. *Let $g : L \to L$ be the endperiodic first return homeomorphism induced on a leaf L of $\mathcal{F}|W$ by a transverse 1-dimensional foliation $\mathcal{L}_g$ of class C^0. If $\varphi_t : L \to L$ is an isotopy of $\varphi_0 = \mathrm{id}$ to a homeomorphism φ_1, then $\mathfrak{C}'_g = \mathfrak{C}'_{\varphi_1 g \varphi_1^{-1}}$.*

Proof. Let N be a closed normal neighborhood of L in W which is a foliated product with leaves the leaves of $\mathcal{F}$ meeting N and normal fibers the arcs of $\mathcal{L} \cap N$. Write $N = L \times [-\varepsilon, \varepsilon]$ and consider each arc ℓ_x of a leaf of $\mathcal{L}$ issuing in the positive direction from $(x, \varepsilon) \in L \times \{\varepsilon\}$ and first returning to N at $(g(x), -\varepsilon)$. In N, replace each arc $\tau_y = \{y\} \times [-\varepsilon, \varepsilon]$ of $\mathcal{L} \cap N$ with an arc $\sigma_y : [-1, 1] \to N$ defined by

$$\sigma_y(t) = (\varphi_{t+1}(y), \varepsilon t),\ -1 \le t \le 0,$$
$$\sigma_y(t) = (\varphi_t^{-1}(\varphi_1(y)), \varepsilon t),\ 0 \le t \le 1.$$

Notice that this still connects $(y, -\varepsilon)$ to (y, ε). We construct an ambient leaf-preserving isotopy ψ, supported in N and carrying each τ_y to σ_y, by

$$\psi_s(y, \varepsilon t) = (\varphi_{s(t+1)}(y), \varepsilon t),\ -1 \le t \le 0,\ 0 \le s \le 1,$$
$$\psi_s(y, \varepsilon t) = (\varphi_{st}^{-1}(\varphi_s(y)), \varepsilon t),\ 0 \le t \le 1,\ 0 \le s \le 1.$$

We obtain $\mathcal{L}'$ from $\mathcal{L}$ by replacing τ_y with σ_y, $\forall y \in L$, observing that the monodromy induced by $\mathcal{L}'$ on $L = L \times \{0\}$ is $\varphi_1 \emptyset g \emptyset \varphi_1^{-1}$. The assertion follows by Proposition 3.11. □

3.7. **Foliated forms.** In this subsection we assume that $g : L \to L$ is a smooth endperiodic monodromy homeomorphism isotopic to f and we choose $\mathcal{L}_g$ to be smooth. In fact (and this is an important remark), all we need is that $\mathcal{L}_g \smallsetminus \mathcal{X}_g$ is smooth and that $\mathcal{L}_g$ itself is of class C^{0+}. Our discussion will be valid for all dimensions.

Definition 3.13. A 1-form $\eta \in A^1(W)$ is a foliated form if it is closed and nowhere vanishing and becomes unbounded at $\partial_\tau \widehat{W}$ in such a way that the corresponding foliation $\mathcal{F}_\eta$ that it defines on W extends to a $C^{\infty,\iota}$ foliation $\widehat{\mathcal{F}}_\eta$ of $\widehat{W}$ by adjunction of $\partial_\tau \widehat{W}$.

We will prove the following.

Theorem 3.14. *The open cone* $\operatorname{int} \mathfrak{C}_g$ *consists of classes in* $H^1(\widehat{W})$ *that can be represented by foliated forms transverse to* $\mathcal{L}_g$. *When* $\mathcal{F}$ *is of class* $C^{\infty,\iota}$, *the interior of* $\mathfrak{C}_g^\kappa$ *is characterized analogously, except that the forms become exact in* $W \smallsetminus K_i$ *for large enough values of* i, *defining a product foliation there identical with* $\mathcal{F}|(W \smallsetminus K_i)$.

Remark that foliated forms only live in W, not in $\widehat{W}$, but $H^1(\widehat{W}) = H^1(W)$ and any form representing a class in W can be taken to be equal to that representing the class in $\widehat{W}$ outside of any small neighborhood of $\partial_\tau \widehat{W}$. If the foliated form is exact in the arms of some octopus decomposition, it is clear that it represents a class in $H^1_\kappa(\widehat{W})$.

Thus, we call $\mathfrak{C}_g$ and $\mathfrak{C}_g^\kappa$ foliation cones associated to g. The rays out of the origin meeting the interior of $\mathfrak{C}_g^\kappa$ correspond to foliations of $\widehat{W}$ that have holonomy only along $\partial_\tau \widehat{W}$ (C^∞ tangent to the identity), are transverse to $\mathcal{L}_g$ and extend $\mathcal{F}|(M \smallsetminus W)$ over M to a foliation of class $C^{\infty,\iota}$, provided $\mathcal{F}$ was $C^{\infty,\iota}$ smooth initially. The rays in $\mathfrak{C}_g$ correspond to foliations (of class $C^{\infty,\iota}$ in $\widehat{W}$) that may only extend $\mathcal{F}|(M \smallsetminus W)$ to a C^{0+} foliation. In both cases, the rational rays correspond to foliations defined by forms η with period group infinite cyclic. These foliations are exactly the ones in which we are interested, being the ones that replace $\mathcal{F}|W$ to give a new finite depth foliation of M. The rest of the rays in $\operatorname{int} \mathfrak{C}_g$ consist of classes having period group dense in $\mathbb{R}$ and so define foliations that are dense leaved in W. Recall that compactness of junctures in C^2 foliations is what forces $\mathcal{F}|\widehat{W}$ to be trivial in the arms when $\mathcal{F}$ is smooth and finite depth.

Proof of Theorem 3.14. Fix a class $[\eta] \in \operatorname{int} \mathfrak{C}_g$, the 1-form $\eta \in [\eta]$ being defined on $\widehat{W}$ and transverse to $\mathcal{X}_g$ (Theorem 3.4). If $\mathcal{F}$ is of class $C^{\infty,\iota}$, select this form to be compactly supported in W, $[\eta] \in \mathfrak{C}_g^\kappa$. Select a neighborhood U of $\mathcal{X}_g$ such that $\eta \pitchfork \mathcal{L}_g|U$. We need to show that η is cohomologous to a foliated form. Note that, if η is compactly supported in K_i, the foliated form will automatically be exact in $W \smallsetminus K_i$.

Given $x \in W \smallsetminus \mathcal{X}_g$, let $s(t)$ be the smooth trajectory along $\mathcal{L}_g$ in that set, smoothly reparametrized so that $x = s(0)$ and $s(\pm 1) \in F_\pm$. Here, F_+ is the

union of outwardly oriented leaves of $\partial_\tau \widehat{W}$ and F_- the union of inwardly oriented ones. For some choices of x both signs may be possible and for others only one. For definiteness, consider the case $s(-1) \in F_-$. Define a tubular neighborhood $V_x = D \times [-1, 3/4)$ of s so that $s(t) = (0, t)$ and $\{z\} \times [-1, 3/4)$ is an arc in $\mathcal{L}_g$, $\forall z \in D$. Here, D is the open unit $(n-1)$-ball with polar coordinates $(r, \theta_1, \dots, \theta_{n-2})$, $0 \le r < 1$. This gives cylindrical coordinates $(t, r, \theta_1, \dots, \theta_{n-2})$ on V_x. On V_x, define a smooth, real valued function

$$\ell_x(t, r, \theta_1, \dots, \theta_{n-1}) = \ell_x(t, r) = \ell(t)\lambda(r),$$

where $\ell(t) = t - 1$, $-1 \le t \le 1/2$, and damps off to 0 smoothly and with positive derivative as $t \to 3/4$, and $\lambda(r) \equiv 1$, $0 \le r \le 1/2$, and damps off to 0 smoothly through positive values as $r \to 1$. Thus, $\ell_x(t, r)$ vanishes outside of V_x and $d\ell_x$ is transverse to $\mathcal{L}_g$ in V_x. Let V_x'ßV_x be the neighborhood of x defined by $-1 \le t < 1/2$ and $0 \le r < 1/2$. Perform an analogous construction for trajectories out of x with $s(1) \in F_+$.

Suitable choices of these open cylinders (using the local compactness) give a locally finite open cover $\{U, V_{x_1}', V_{x_2}', \dots\}$ of $\widehat{W}$. For suitable choices of positive constants c_i, set $\ell = \sum_{i=1}^{\infty} c_i \ell_{x_i}$, a smooth function, supported in $W \setminus \mathcal{X}_g$, with $d\ell \pitchfork \mathcal{L}_g$ outside of a compact neighborhood of $\mathcal{X}_g$ in U. Since η is bounded in any compact region of $\widehat{W}$ and is transverse to $\mathcal{L}_g$ in U, we can choose the coefficients $c_i > 0$ large enough that $\eta' = \eta + d\ell$ is a closed form in $\widehat{W}$, cohomologous to η and transverse to $\mathcal{L}_g$. This form might be badly behaved at $\partial_\tau \widehat{W}$, hence we must modify it by adding on a suitable exact form supported in a neighborhood of the boundary leaves.

Let $V = F_- \times [0, 1)$ be a normal neighborhood of F_- in $\widehat{W}$, the fibers being arcs in leaves of $\mathcal{L}_g$. Let λ be a smooth function on the deleted normal neighborhood $V \setminus F_-$, depending only on the normal parameter t, and having $\lambda'(t) \ge 0$, with $\lambda'(t) = e^{1/t^2}$ near F_-. We claim that $\widetilde{\eta} = \eta' + d\lambda$ is the desired foliated form in the cohomology class $[\eta]$. Indeed, it is everywhere transverse to $\mathcal{L}_g$, hence nonsingular on W, it is cohomologous to η' and it becomes unbounded at $\partial_\tau \widehat{W}$. We must show that $\ker \widetilde{\eta}$ extends $C^{\infty,\iota}$-smoothly to a plane field on $\widehat{W}$ by adding on the tangent planes to $\partial_\tau \widehat{W}$. For this, set $\overline{\eta} = \widetilde{\eta}/\lambda' = \eta'/\lambda' + dt$, a form defined on a small enough deleted neighborhood of F_-. This form is no longer closed but satisfies $\ker \overline{\eta} = \ker \widetilde{\eta}$ in that neighborhood. Since η' is bounded on $\widehat{W}$, it is clear that $\overline{\eta}$ approaches dt in the C^∞ topology as $t \to 0$ and that the resulting foliation of $\widehat{W}$ is of class $C^{\infty,\iota}$. After a similar construction in a normal neighborhood of F_+, we obtain a foliated form, again denoted by $\widetilde{\eta}$, transverse to $\mathcal{L}_g$.

While all of this works equally well whether or not η is compactly supported in W, in the case that $\mathcal{F}$ is of class $C^{\infty,\iota}$, we choose η compactly supported and want to produce the foliated form $\widetilde{\eta}$ so that, outside of some compact region in $\widehat{W}$, it agrees with the (exact) foliated form ω that defined $\mathcal{F}$ there. For i large, $\omega|(\widehat{W} \setminus K_i) = d\gamma$ and $\operatorname{supp} \eta$ß$K_i^{ø}$. Choose $r > 0$

so that $\{U, V'_{x_1}, V'_{x_2}, \dots, V'_{x_r}\}$ covers K_{i+1}. Smoothly damp γ off to 0 in a neighborhood N of K_i in K_{i+1} so that it becomes 0 near K_i and is unchanged in $\widehat{W} \smallsetminus N$, extending it by 0 over K_i. Call this new function $\widetilde{\gamma}$. Similarly, damp λ off to 0 in $V \smallsetminus K_{i+1}$ so that $d\lambda$ remains transverse to $\mathcal{L}_g$ in $V \smallsetminus K_{i+1}$ and becomes 0 in $V \smallsetminus K_{i+2}$, calling this new function $\widetilde{\lambda}$. Now, $\widetilde{\eta} = \eta + \sum_{j=1}^{r} c_j d\lambda_{x_j} + d\widetilde{\lambda} + d\widetilde{\gamma}$ is as desired for suitably large choices of the constants c_j. The proof of Theorem 3.14 is complete. $\square$

Remark. In the case that $\mathcal{F}$ is of class $C^{\infty,\iota}$, refoliating W by the new foliated form $\widetilde{\eta}$ as above only changes $\mathcal{F}$ in a compact part of $\widehat{W}$. Thus, the *germinal* holonomy along all leaves of $\mathcal{F}|(M \smallsetminus \widehat{W})$ has been unchanged, while the infinitesimal holonomy along the leaves of $\partial_\tau \widehat{W}$ is C^∞-tangent to the identity. Thus, the new foliation of M is again smooth of class $C^{\infty,\iota}$. Refoliating finitely many components of $\mathcal{O}$ in this way continues to preserve this smoothness class. If one wants to simultaneously refoliate *all* components of $\mathcal{O}$, preserving smoothness gets a bit delicate. Instead of tackling this here, we remark that the new foliation, of class C^{0+} and finite depth, satisfies the hypotheses of [10], hence is homeomorphic to a foliation of class $C^{\infty,\iota}$.

4. The Handel-Miller cones

In this section, we restrict our attention to the case that $\mathcal{F}$ is smooth and $\dim M = 3$. Consequently, the monodromy $f : L \to L$ is endperiodic [4, Proposition 2.16].

For a detailed treatment of the Handel-Miller theory of endperiodic maps of surfaces, see [4]. Notation will be based on that paper save mention to the contrary.

Let $h : L \to L$ be the Handel-Miller representative of the isotopy class of the monodromy f of L. We will show that the cone $\mathfrak{C}_h^\kappa$ is the maximal foliation cone corresponding to that isotopy class. One difficulty is that "the" Handel-Miller representative is a misnomer. There are infinitely many, but we will show that $\mathfrak{C}_h^\kappa$ is independent of the choice. For this, we need to show that the homology cone $\mathfrak{C}'_h$, hereafter called the *Handel-Miller cone*, is independent of the choice. We need to investigate the asymptotic cycles for $\mathcal{X}_h$ more carefully.

Generally, the choices of h are not smooth, perhaps not even regular, hence the interpretation of $\mathfrak{C}_h^\kappa$ as a foliation cone seems problematic. However, by [4, Theorem 12.1], there is a choice of h which has associated $\mathcal{L}_h$ satisfying the hypotheses of Subsection 3.7.

4.1. **The invariant set.** For the time being, no smoothness hypotheses are imposed on h nor on $\mathcal{L}_h$. The lamination $\mathcal{X}_h$ is the $\mathcal{L}_h$-saturation of the invariant set $X_h = L \smallsetminus (\mathcal{U}_+ \cup \mathcal{U}_-)$, the set of points that do not escape to ends of L under forward or backward iteration of h. We recall that h leaves invariant a pair of pseudo-geodesic laminations $\Lambda_\pm$ and that $X_h^* =$

$\Lambda_+ \cap \Lambda_- \subseteq X_h$ is called the *meager* invariant set. Generally, $X_h^* \neq X_h$. Referring to [4] (where X_h was denoted by $\mathcal{I}$ and X_h^* by $\mathcal{K}$), we describe the set $X_h \smallsetminus X_h^*$.

The complement $L \smallsetminus (\Lambda_+ \cup \Lambda_-)$ has infinitely many components, "most" of which lie in the $\pm$-escaping set $\mathcal{U}_+ \cup \mathcal{U}_-$, hence are disjoint from X_h. Those components that do not lie in the $\pm$-escaping set are the nuclei N_i of finitely many principal regions P_i, $1 \leq i \leq n$. The nucleus N_i is a compact, connected surface meeting Z_h^* in finitely many vertices [4, Lemma 6.40]. The principal regions and their nuclei are permuted amongst themeselves by h. For a complete treatment of principal regions and their nuclei, see [4, Subsection 6.5].

Thus, if we set $N = N_1 \cup \cdots \cup N_n$, we can write $X_h = X_h^* \cup N$. If N_i is simply connected, $h|N_i$ can be defined so that $h^p|N_i$ is the identity, for a minimal integer $p \geq 1$. The $\mathcal{L}_h$-saturation of such N_i contributes only one ray to the Handel-Miller cone $\mathfrak{C}_h'$, spanned by the closed orbit of any vertex of ∂N_i. Thus, this ray already was contributed by the $\mathcal{L}_h$-saturation of X_h^*. If not simply connected, N_i is the union of closed annuli $A_{i,1}, \dots, A_{i,r_i}$ and a "core" N_i'. The union of the annuli, taken over all admissible index pairs (i,j), is h-invariant and each $A_{i,j}$ is cobounded by a polygonal circle $s_{i,j}$, with edges lying alternately in leaves of Λ_+ and Λ_-, and another simple closed reducing curve $\rho_{i,j}$. These reducing curves constitute the boundary of N_i'. If $p > 0$ is an integer such that $h^p(A_{i,j}) = A_{i,j}$ and h^p fixes each vertex of $s_{i,j}$, then $h^p|A_{i,j}$ is isotopic to the identity. In the $\mathcal{L}_h$-saturation of the annuli $A_{i,j}$, it is evident that the homology directions are homologous to positive multiples of the closed orbits of the vertices of the polygonal circles $s_{i,j}$. These vertices lie in X_h^*. Thus, the $\mathcal{L}_h$-saturations of the annuli contribute no classes to $\mathfrak{C}_h'$ other than those contributed by the $\mathcal{L}_h$-saturation $\mathcal{X}_h^* ß \mathcal{X}_h$ of X_h^*. All new classes in the Handel-Miller cone will be contributed only by homology directions of the $\mathcal{X}_h$-saturation of the cores N_i'.

There is a power h^p, for a minimal integer $p \geq 1$, that maps each core N_i' into itself and is the identity on the set $N \cap X_h^*$ of vertices. By the Nielsen-Thurston theory [1, 17, 24], there is a family of simple closed curves (reducing circles) in N_i', splitting that surface into subsurfaces in each of which some power h^{kp} is either isotopic to a pseudo-Anosov homeomorphism or a periodic homeomorphism. As part of our definition of "Handel-Miller homeomorphism" we will require that h^{kp} be exactly pseudo-Anosov or periodic on each of these pieces. (We will say that $h|N$ is of Nielsen-Thurston type.) Actually, the reducing circles need to be slightly thickened to invariant annuli. These annuli contribute no cycles to the cone not already contributed by the periodic and/or pseudo-Anosov pieces. Each periodic piece contributes a ray of homology classes to $\mathfrak{C}_h'$. Each pseudo-Anosov piece contributes a closed, convex subcone. Although the choice of $h|N$ is not unique, any two choices h and h' are related by $(h')^{kp} = \varphi ø h^{kp} ø \varphi^{-1}$,

where $\varphi : L \to L$ is a homeomorphism isotopic to the identity and supported on N (cf. [35, page 421]). By Corollary 3.12, the Handel-Miller cone $\mathfrak{C}'_h = \mathfrak{C}'_{h^{kp}}$ is independent of the allowable choices of $h|N$.

4.2. **Uniqueness of the Handel-Miller cone.** The laminations $\Lambda_\pm$ associated to a Handel-Miller homeomorphism h are augmented by the h-junctures [4] to give mutually transverse, h-invariant laminations $\Gamma_\pm$. Let $Y_h = \Gamma_+ \cap \Gamma_-$. Each arc of $\Gamma_+ \cup \Gamma_- \smallsetminus Y_h$ completes to a compact arc meeting Y_h only in its endpoints. If we replace each such arc with the unique homotopic geodesic joining the endpoints, we obtain new laminations $\Gamma^\flat_\pm$. Note that $\Gamma^\flat_+ \cap \Gamma^\flat_- = Y_h$. We can now extend $h|Y_h$ to a Handel-Miller endperiodic automorphism $h^\flat : L \to L$ by essentially the same procedure as in the proof of [4, Theorem 8.1], the laminations $\Gamma^\flat_\pm$ being the extended laminations for $h^\flat$.

In defining $h^\flat|N^\flat$ by a Nielsen-Thurston homeomorphism, there is a certain latitude. One first finds a homeomorphism $\psi : L \to L$, isotopic to the identity and carrying ∂N to $\partial N^\flat$ [16]. Then $\psi : N \to N^\flat$ is isotopic to the inclusion. We set $h^\flat|N^\flat = \psi \o h|N \o \psi^{-1}$. By the discussion in the previous subsection and the fact that h and $h^\flat$ agree on $X^*_h = X^*_{h^\flat}$, we see that $\mathfrak{C}'_h = \mathfrak{C}'_{h^\flat}$.

If g is another Handel-Miller representative of the isotopy class of h, we produce $g^\flat$ as above and corresponding extended laminations $\Gamma^{g^\flat}_\pm$. Similarly, the extended laminations for $h^\flat$ will be denoted by $\Gamma^{h^\flat}_\pm$. Since $h^\flat$ and $g^\flat$ are isotopic, they have lifts to the universal cover of L which induce the same homeomorphisms on the ideal boundary at infinity. Thus, in standard fashion (cf. [4, Theorem 8.1]), there is a natural homeomorphism $\theta : L \to L$, isotopic to the identity, that carries $\Gamma^{g^\flat}_\pm$ to $\Gamma^{h^\flat}_\pm$. Then $\theta^{-1} \o h^\flat \o \theta$ is a legitimate choice of $g^\flat$. Thus, appealing to Corollary 3.12, we get

$$\mathfrak{C}'_h = \mathfrak{C}'_{h^\flat} = \mathfrak{C}'_{\theta^{-1} \o h^\flat \o \theta} = \mathfrak{C}'_{g^\flat} = \mathfrak{C}'_g.$$

We have completed the proof of the following.

Theorem 4.1. *The Handel-Miller cone $\mathfrak{C}'_h \ss H_1(\widehat{W})$ is independent of the choice of the Handel-Miller representative of the isotopy class of the endperiodic monodromy of L.*

Remark. Because of this theorem, we will denote the cone $\mathfrak{C}'_h$ by $\mathfrak{C}'_{\mathcal{F}}$ and the dual $H^1_\kappa(\widehat{W})$-cohomology cone by $\mathfrak{C}^\kappa_{\mathcal{F}}$. We will call this latter the *Handel-Miller foliation cone.*

4.3. **Handel-Miller foliation cones are polyhedral.** We fix a Handel-Miller monodromy map $h : L \to L$. By [4, Theorem 12.1], this will now be chosen to be a diffeomorphism except at the finitely many p-pronged singularities in the interior of pseudo-Anosov components (if any) of N. Then $\mathcal{L}_h$ can be chosen to be smooth except, perhaps, at finitely many

closed orbits corresponding to the p-pronged singularities. Standard models of the neighborhoods of these orbits (cf. [13, Appendix B]) make it easy to arrange that $\mathcal{L}_h$ is of class C^{0+} at those closed orbits.

As in [4, Section 9], the dynamical system $h : X_h^* \to X_h^*$ is conjugate to a 2-ended subshift of finite type. As is standard, on the pseudo-Anosov pieces in N', the action of h is semi-conjugate to such a 2-ended subshift, where we use new letters. On the periodic pieces, only an orbit of minimal period is needed to contribute to $\mathfrak{C}'_{\mathcal{F}}$. This orbit is encoded, up to a shift, by a periodic, bi-infinite sequence composed of new letters, and the action of h on that orbit is conjugate to the shift. Thus the closed sublamination $\mathfrak{X}_h^\bullet$ of $\mathfrak{X}_h$, given by the saturations of X_h^*, of each pseudo-Anosov piece in N', and of a single orbit from each periodic piece in N', carries all the asymptotic cycles needed to span $\mathfrak{C}'_{\mathcal{F}}$ and corresponds to an h-invariant subset $X_h^\bullet$ßX_h on which h is semi-conjugate to a 2-ended subshift of finite type. Write this subshift as $\sigma_A : SS_A \to SS_A$, where SS_A is a collection of bi-infinite sequences $\iota = (i_k)_{k\in\mathbb{Z}}$ of finitely many letters, A is a matrix of 0's and 1's encoding which letter can follow which, and σ_A shifts each letter one position to the right. The finitely many distinct letters i each label a rectangle R_ißL of a Markov partition. (As in [4, Section 9], we allow degenerate rectangles. Those corresponding to a periodic piece in N' degenerate to a singleton.) These rectangles cover $X_h^\bullet$ and do not overlap. An element $\iota = (i_k)_{k\in\mathbb{Z}} \in SS_A$ represents a unique point $x_\iota \in R_{i_0} \cap X_h^\bullet$ such that $h^k(x_\iota) \in R_{i_k}$, $\forall k \in \mathbb{Z}$. In terms of the lamination $\mathfrak{X}_h^\bullet$, this means that the leaf issuing from x_ι meets L successively in $R_{i_0}, R_{i_1}, \dots, R_{i_k}, \dots$ in forward time, with a corresponding statement for backward time. Points in X_h^* have a unique representative sequence $\iota \in SS_A$ [4, Section 9], but some points in the pseudo-Anosov pieces of N' may have finitely many such representatives. The problem is that distinct Markov rectangles in N' may meet along parts of their boundaries. Thus the map $\iota \mapsto x_\iota$ is finite to one, defining a semi-conjugacy of σ_A to $h|X_h^\bullet$.

The periodic elements of SS_A are those carried to themselves by some power σ_A^q, $q \geq 1$. These correspond to closed leaves in $\mathfrak{X}_h^\bullet$. The substring $(i_0, i_1, \dots, i_{q-1})$ of a periodic sequence ι, $\sigma_A^q(\iota) = \iota$, where $q \geq 1$ is minimal, will be called the period of ι. The substring $(i_0, i_1, \dots, i_{q-1}, i_0)$ will be called a periodic string. If no proper substring of a period is a periodic string, we say that the period is *minimal.* Since there are only finitely many distinct entries occurring in the sequences $\iota \in SS_A$, it is evident that there are only finitely many minimal periods. Those closed leaves γ of $\mathfrak{X}_h^\bullet$ that correspond to minimal periods in the symbolic system will be called minimal loops in $\mathfrak{X}_h^\bullet$ and denoted by $\gamma_1, \gamma_2, \dots, \gamma_r$. Our goal is to prove the following.

Theorem 4.2. *The Handel-Miller cone $\mathfrak{C}'_{\mathcal{F}}$ is the convex hull of the union of rays*

$$\mathbb{R}^+[\gamma_1] \cup \mathbb{R}^+[\gamma_2] \cup \cdots \cup \mathbb{R}^+[\gamma_r].$$

Thus, the dual cone $\mathfrak{C}^{\kappa}_{\mathcal{F}}$ is polyhedral, defined by the linear inequalities $[\gamma_i] \geq 0$, $1 \leq i \leq r$, *hence has only finitely many faces.*

Let Φ_t denote the flow on $\widehat{W}$ that stabilizes $\partial_\tau \widehat{W}$ pointwise, has flow lines in W coinciding with the leaves of $\mathcal{L}_h$ and is parametrized so as to preserve $\mathcal{F}|W$ and so that $\Phi_1|L = h$.

Let $\iota = (i_k)_{k=-\infty}^{\infty} \in SS_A$ and suppose that $i_q = i_0$ for some $q > 0$. Let $x \in R_\iota = \bigcap_{j=-\infty}^{\infty} R_{i_j}$. Then there is a corresponding singular cycle Γ_q formed from the orbit segment $\gamma_q = \{\Phi_t(x)\}_{0 \leq t \leq q}$ and an arc $\tau ß R_{i_0}$ from $\Phi_q(x) = h^q(x)$ to x. Also, since $i_q = i_0$, there is a periodic element $\iota' \in \Sigma_A$ with period $i_0, \dots, i_{q-1}$ and a corresponding closed leaf $\Gamma_{\iota'} = \Gamma'$ of $\mathcal{X}_h^\bullet$.

Lemma 4.3. *The singular cycle Γ_q and closed leaf Γ', obtained as above, are homologous in $\widehat{W}$. In particular, the homology class of Γ_q depends only on the periodic element ι'.*

Proof. The loop Γ' is the orbit segment $\{\Phi_t(y)\}_{0 \leq t \leq q}$, for a periodic point

$$y \in R_{i_0} \cap h^{-1}(R_{i_1}) \cap \cdots \cap h^{-q}(R_{i_q}) = R'.$$

Remark that $x \in R'$ also. Let τ' be an arc in the rectangle R' from x to y and set $\tau'' = h^q(\tau')$, an arc in $h^q(R')$ from $h^q(x)$ to y. Since $i_q = i_0$, $h^q(R') ß R_{i_0}$ and the cycle $\tau + \tau' - \tau''$ in the rectangle R_{i_0} is homologous to 0. That is, we can replace the cycle $\Gamma_q = \gamma_q + \tau$ by the homologous cycle $\gamma_q - \tau' + \tau''$. Finally, a homology between this cycle and Γ' is given by the map

$$H : [0,1] \times [0,q] \to \widehat{W},$$

defined by parametrizing τ' on $[0,1]$ and setting

$$H(s,t) = \Phi_t(\tau'(s)).$$

□

Corollary 4.4. *Every closed leaf Γ of $\mathcal{X}_h^\bullet$ is homologous in $\widehat{W}$ to a linear combination of the minimal loops in $\mathcal{X}_h^\bullet$ with non–negative integer coefficients.*

Proof. The closed leaf Γ corresponds to a period $(i_0, \dots, i_{q-1})$. If this period is minimal, we are done. Otherwise, after a cyclic permutation, we can assume that the period is of the form $(i_0, i_1, \dots, i_p = i_0, i_{p+1}, \dots, i_{q-1})$. We then see that Γ is homologous to the sum of two loops, one being the arc γ of Γ corresponding to the periodic string $(i_0, \dots, i_p = i_0)$ followed by an arc τ from the endpoint of γ to its initial point, and one being $-\tau + \gamma'$, where γ' is the subarc of Γ corresponding to the periodic string $(i_0, i_{p+1}, \dots, i_{q-1}, i_q = i_0)$. By Lemma 4.3, both $\gamma + \tau$ and $-\tau + \gamma'$ are homologous to closed orbits corresponding to periods strictly shorter than $(i_0, \dots, i_{q-1})$. Thus, finite iteration of this procedure proves the corollary. □

Let $\iota = (i_k)_{k=-\infty}^{\infty}, \iota' = (i'_k)_{k=-\infty}^{\infty} \in SS_A$ and suppose that

$$(i_0, i_1, \dots, i_q) = (i'_0, i'_1, \dots, i'_q),$$

not necessarily a period. Let $x = x_\iota$ and $x' = x_{\iota'}$. Both of these points are in

$$R' = R_{i_0} \cap h^{-1}(R_{i_1}) \cap \cdots \cap h^{-q}(R_{i_q}).$$

Choose a path τ in R_{i_0} from x' to x and a path τ'ßR_{i_q} from $h^q(x')$ to $h^q(x)$. Consider the orbit segments $\Gamma = \{\Phi_t(x)\}_{t=0}^{q}$ and $\Gamma' = \{\Phi_t(x')\}_{t=0}^{q}$. Let K be an upper bound of the diameters of R_i, $1 \le i \le n$. Then the paths τ and τ' can always be chosen to have length less than K.

The following is proven analogously to Lemma 4.3.

Lemma 4.5. *The singular chains Γ' and $\tau + \Gamma - \tau'$ are homologous. In particular, for each closed 1-form η on $\widehat{W}$,*

$$\int_{\Gamma'} \eta = \int_{\tau+\Gamma-\tau'} \eta.$$

Here, the paths τ and τ' have length less than K.

Proposition 4.6. *Every homology direction can be arbitrarily well approximated by nonnegative linear combinations of the minimal loops.*

Proof. Let $\Gamma = \{\Phi(t)(x)\}_{t=-\infty}^{\infty}$ be an orbit and suppose that x corresponds to the symbol $\iota = \{i_r\}_{r=-\infty}^{\infty}$. By a suitable shift, we can assume that i_0 occurs infinitely often in forward time in this symbol. Consequently, for each index i in ι, there is a positive integer k_i such the the (i, i_0)-entry in A^{k_i} is strictly positive. Let k be the largest of the k_i. Thus, given a substring $(i_0, i_1, \dots, i_q)$ of ι, there is a periodic element $\iota' \in \Sigma_A$ with period $(i_0, i_1, \dots, i_q, i_{q+1}, \dots, i_{q+s})$, where $s \le k$. Let Γ'_q denote the corresponding periodic orbit. If we parametrize the flow Φ_t by the invariant measure for $\mathcal{F}$ of period 1, then the length of the segment Γ_q of Γ corresponding to the string $(i_0, i_1, \dots, i_q)$ is q. Choosing a suitable sequence $q \uparrow \infty$, we obtain the general homology direction

$$\mu = \lim_{q\to\infty} \frac{1}{q} \int_{\Gamma_q} .$$

Passing to a subsequence, we also obtain a cycle

$$\mu' = \lim_{q\to\infty} \frac{1}{q} \int_{\Gamma'_q} .$$

Since s is bounded independently of q, Lemma 4.5 implies that μ and μ' agree on all closed 1-forms, and so μ' is a cycle homologous to μ (both in $(\mathcal{D}'_*, \partial)$ and $(\mathcal{E}'_*, \partial)$). Corollary 4.4 then implies the assertion. $\square$

By Lemma 3.7, Theorem 4.2 follows

4.4. **Maximality of the Handel-Miller cones.** Our next goal is to show that, if g is an endperiodic map in the isotopy class of h, then $\mathfrak{C}_g^\kappa \subseteq \mathfrak{C}_{\mathcal{F}}^\kappa$. In fact, we will show that if g is a monodromy map for *any* fibration $\mathcal{G}$ of W that appropriately extends $\mathcal{F}|(M \smallsetminus W)$, then either $(\operatorname{int} \mathfrak{C}_g^\kappa) \cap \mathfrak{C}_{\mathcal{F}}^\kappa = \emptyset$, or $\mathfrak{C}_g^\kappa \subseteq \mathfrak{C}_{\mathcal{F}}^\kappa$.

We begin with an analysis of the group $G = H^1_\kappa(\widehat{W}) \cap H^1(\widehat{W};\mathbb{Z})$. This will play the role of the integer lattice in $H^1_\kappa(\widehat{W})$. The "rational rays" in this vector space will be those rays issuing from the origin that meet G in nonzero points. It will be crucial that the union of the rational rays be dense in $H^1_\kappa(\widehat{W})$.

Recall from Subsection 3.2 that the topology on $H^1_\kappa(\widehat{W}) = \varinjlim H^1_\kappa(K_i^{\varnothing})$ can be taken to be the weak topology.

By ∂K_i we will mean the relative boundary of K_i in $\widehat{W}$. It consists of finitely many disjoint rectangles and annuli. No component can be a Möbius strip since it will be fibered by *oriented* intervals that are arcs of $\mathcal{L}_h$.

Lemma 4.7. *The subspace $H^1_\kappa(K_i^{\varnothing})$ß$H^1(K_i)$ consists of those classes which restrict to* 0 *in $H^1(\partial K_i)$.*

Proof. If $[\omega] \in H^1(K_i)$ restricts to 0 on ∂K_i, then $\omega|\partial K_i$ is exact, hence is also exact in a normal neighborhood N of ∂K_i. That is, $\omega|N = d\lambda$ for a smooth function λ which can be damped off to 0 and extended by 0 to a smooth function λ on all of K_i. Then $\omega - d\lambda$ has compact support in $K_i^{\varnothing}$. The converse is immediate. □

Set $G_i = H^1_\kappa(K_i^{\varnothing}) \cap H^1(K_i;\mathbb{Z})$, a finitely generated, free abelian group.

Lemma 4.8. *The subgroup G_iß$H^1_\kappa(K_i^{\varnothing})$ is a full lattice subgroup of* (*i.e., spans*) *this finite dimensional vector space.*

Proof. The core loops of the annular components of ∂K_i represent elements (not necessarily linearly independent) of the finitely generated, free abelian group $\Gamma_i = H_1(K_i;\mathbb{Z})/\,$torsion. We call them "peripheral" elements. Let $\Gamma_i' \subseteq \Gamma_i$ be the smallest direct summand in this lattice that contains the peripheral elements. In particular, by Lemma 4.7, the classes in $H^1(K_i)$ which vanish on Γ_i' are exactly the elements of $H^1_\kappa(K_i^{\varnothing})$. If $[\sigma_1], [\sigma_2], \dots, [\sigma_r]$ are a basis of Γ_i', then extend to an ordered basis $([\sigma_1], \dots, [\sigma_r], [\sigma_{r+1}], \dots, [\sigma_q])$ of Γ_i, hence of $H_1(K_i)$, Let $([\omega_1], \dots, [\omega_r)], [\omega_{r+1}], \dots, [\omega_q])$ be the dual basis of $H^1(K_i)$. Then, by Lemma 4.7, the classes $[\omega_j]$ lie in G_i, $r+1 \le j \le q$, and form a basis of $H^1_\kappa(K_i^{\varnothing})$. □

It is standard that the rays in the finite dimensional vector space $H^1_\kappa(K_i^{\varnothing})$ which meet the full lattice G_i at nonzero points (the rational rays) unite to form a dense subset of that vector space. Furthermore, note that $G_0 \subseteq G_1 \subseteq \cdots \subseteq G_i \subseteq \cdots$ has increasing union Gß$H^1_\kappa(\widehat{W})$. An appeal to the definition of the weak topology makes the following clear.

Corollary 4.9. *The union of the rational rays is dense in $H^1_\kappa(\widehat{W})$.*

Proposition 4.10. *The proper foliated ray* $\langle \mathcal{G} \rangle$ *lies in* $\operatorname{int} \mathfrak{C}^\kappa_{\mathcal{F}}$ *if and only if* $\mathfrak{C}^\kappa_{\mathcal{G}} = \mathfrak{C}^\kappa_{\mathcal{F}}$.

Proof. Suppose that $\langle \mathcal{G} \rangle$ ß $\operatorname{int} \mathfrak{C}^\kappa_{\mathcal{F}}$. By Theorem 3.14, we take the foliation $\mathcal{G}$ to be transverse to $\mathcal{L}_h$. By the "transfer theorem" [4, Theorem 11.1], $\mathcal{L}_h$ induces Handel-Miller monodromy g on each leaf of $\mathcal{G}|W$. A comment is needed since, in [4], it was not required that Handel-Miller monodromy be exactly Nielsen-Thurston on N'. But, as Fried [18] proves the transfer theorem for the Nielsen-Thurston pseudo-Anosov pieces, the discussion in Subsection 11.4 of [4] is easily augmented to accomodate that requirement. Thus, the cones $\mathfrak{C}^\kappa_{\mathcal{G}}$ and $\mathfrak{C}^\kappa_{\mathcal{F}}$ are determined by the same core lamination $\mathcal{X}_g = \mathcal{X}_h$ and so are identical. For the converse, $\mathfrak{C}^\kappa_{\mathcal{G}} = \mathfrak{C}^\kappa_{\mathcal{F}}$, for proper foliated rays $\langle \mathcal{G} \rangle$ and $\langle \mathcal{F} \rangle$, clearly implies that $\langle \mathcal{G} \rangle$ ß $\operatorname{int} \mathfrak{C}^\kappa_{\mathcal{F}}$. □

Corollary 4.11. *No proper foliated ray is contained in* $\partial \mathfrak{C}^\kappa_{\mathcal{F}}$.

Proof. If there is a proper foliated ray $\langle \mathcal{G} \rangle$ ß$\partial \mathfrak{C}^\kappa_{\mathcal{F}}$, then $\operatorname{int} \mathfrak{C}^\kappa_{\mathcal{G}} \cap \operatorname{int} \mathfrak{C}^\kappa_{\mathcal{F}} \neq \emptyset$. By Corollary 4.9, there is a proper foliated ray $\langle \mathcal{H} \rangle$ ß $\operatorname{int} \mathfrak{C}^\kappa_{\mathcal{G}} \cap \operatorname{int} \mathfrak{C}^\kappa_{\mathcal{F}}$. By Proposition 4.10, we see that $\mathfrak{C}^\kappa_{\mathcal{G}} = \mathfrak{C}^\kappa_{\mathcal{H}} = \mathfrak{C}^\kappa_{\mathcal{F}}$. That is, $\langle \mathcal{G} \rangle$ ß $\operatorname{int} \mathfrak{C}^\kappa_{\mathcal{F}}$, contrary to our hypothesis. □

The boundary $\partial \mathfrak{C}^\kappa_{\mathcal{F}}$ is made up of r top faces $F_1, \dots, F_r$, where F_i is a convex, polyhedral cone with nonempty (relative) interior in the hyperplane $[\gamma_i] = 0$.

Lemma 4.12. *Each* F_i *contains a dense family of rays that meet points of the integer lattice* $H^1_\kappa(\widehat{W}; \mathbb{Z})$.

Proof. Since $[\gamma_i]$ is an integral homology class, it takes integer values on the integer lattice. The corresponding claim is standard in the finite dimensional spaces $H^1_\kappa(K^\phi_j)$ and remains true in the direct limit by the definition of the weak topology. □

Theorem 4.13. *If* g *is a monodromy map* (*endperiodic*) *for a fibration* $\mathcal{G}$ *of* W *that appropriately extends* $\mathcal{F}|(M \smallsetminus W)$, *then either* $(\operatorname{int} \mathfrak{C}^\kappa_g) \cap \mathfrak{C}^\kappa_{\mathcal{F}} = \emptyset$, *or* $\mathfrak{C}^\kappa_g \subseteq \mathfrak{C}^\kappa_{\mathcal{F}}$. *In particular,* $\mathfrak{C}^\kappa_{\mathcal{F}} = \mathfrak{C}^\kappa_h$ *is the maximal foliation cone for monodromies in the isotopy class of* h.

Proof. If $(\operatorname{int} \mathfrak{C}^\kappa_g) \cap \mathfrak{C}^\kappa_{\mathcal{F}} \neq \emptyset$ and $\mathfrak{C}^\kappa_g \not\subseteq \mathfrak{C}^\kappa_{\mathcal{F}}$, then Lemma 4.12 implies that there is a proper foliated ray in $\partial \mathfrak{C}^\kappa_{\mathcal{F}}$, contradicting Corollary 4.11. □

Remark. Correspondingly, the dual homology cone $\mathfrak{C}'_{\mathcal{F}} = \mathfrak{C}'_h$ is the minimal $\mathfrak{C}'_g$ for all monodromies g isotopic to h. In this sense, we can say that the Handel-Miller monodromy has the "tightest" dynamics in its isotopy class.

4.5. **Foliated products.** We have been assuming that W is not a foliated product. If $\widehat{W} = F \times I$, the cohomological classification of the foliations is quite trivial. Given any closed, compactly supported 1-form η on $\widehat{W}$, one modifies it to a foliated form, transverse to the interval fibers, as in Section 3.7. In this case, one can apply the construction to the 1-form 0,

producing an exact foliated form dg. The class $0 \in H^1_\kappa(\widehat{W})$ represents the product foliation. As remarked in the introduction, we regard $H^1_\kappa(\widehat{W})$ itself as a foliation cone and $\{0\}$ as a degenerate proper foliated ray.

4.6. **Uniqueness of the foliations up to isotopy.** Let $\mathcal{F}$ and $\mathcal{H}$ be foliations of $\widehat{W}$ which fiber W and appropriately extend $\mathcal{F}|(M \smallsetminus W)$ and suppose that $\langle \mathcal{F} \rangle = \langle \mathcal{H} \rangle$. These two foliations are trivial outside K_i, for large enough i. Both $\mathcal{F}$ and $\mathcal{H}$ restrict to depth one foliations of the sutured manifold K_i that are defined by cohomologous foliated forms there. By [9], these foliations are isotopic in K_i and it is easy to make this isotopy global in $\widehat{W}$. (In the case of foliated products, it should be remarked that the fact that the foliations classified by 0 are all isotopic to the product foliation is, in fact, quite deep. It is true in each $K_{j+1} \smallsetminus K_j^\varnothing$, $i \leq j < \infty$ by the Laudenbach-Blank theorem [27] or, equivalently, by a theorem of J. Cerf [14]. It is then propagated to all of $\widehat{W}$ as $i \to \infty$.) The isotopy is smooth in W, but not at $\partial_\tau W$.

4.7. **Finiteness of the set of Handel-Miller cones.** The nucleus K_0 of the octopus decomposition of $\widehat{W}$ cuts off finitely many arms which are of the form $B \times I$, where $B \text{ß} \partial_\tau \widehat{W}$ is connected and noncompact. All of our foliations $\mathcal{G}$ that appropriately extend $\mathcal{F}|(M \smallsetminus W)$, when restricted to an arm, are transverse to the I fibers. Thus, for all the appropriate foliations $\mathcal{G}$ with Handel-Miller monodromy g, the core lamination $\mathcal{X}_g$ lies in K_0. The cone $\mathfrak{C}^\kappa_{\mathcal{G}}$ is defined by inequalities $[\gamma_i] \geq 0$, $1 \leq i \leq r$, where each γ_i is a periodic orbit in $\mathcal{X}_g$. These same inequalities define the cone $\mathfrak{C}_{\mathcal{G}|K_0} \text{ß} H^1(K_0)$ determined by the depth one foliation $\mathcal{G}|K_0$ of the sutured manifold K_0. This sets up a one-to-one correspondence between the set $\mathcal{K}_W$ of foliation cones in $H^1_\kappa(\widehat{W})$ and a subset of the set of foliation cones in $H^1(K_0)$. By [11, Theorem 6.4], it follows that there are only finitely many of these cones.

The proof of Theorem 1.4 for foliations of class $C^{\infty,\iota}$ and the rational rays is complete.

5. Foliations of class C^{0+}

The isomorphism $H^1(\widehat{W}) = \varprojlim H^1(K_i)$ is induced by the natural homomorphisms $\psi_i : H^1(\widehat{W}) \to H^1(K_i)$. Recall from Subsection 3.2 that the topology on $H^1(\widehat{W})$ is the standard inverse limit topology, relativized from the Tychonov topology via the inclusion

$$\varprojlim H^1(K_i) \text{ß} H^1(K_0) \times H^1(K_1) \times \cdots \times H^1(K_i) \times \cdots .$$

Since ψ_i is induced by projection of the product onto its ith factor, it is continuous.

Let $V_i \text{ß} H^1(K_i)$ denote the image of ψ_i and remark that $H^1(\widehat{W}) = \varprojlim V_i$. Also note that, since $H^1(\widehat{W}) = H^1(\widehat{W}; \mathbb{Z}) \otimes \mathbb{R}$, the integer lattice $H^1(\widehat{W}; \mathbb{Z})$ is carried onto a full lattice subgroup of V_i, $i \geq 0$. Indeed, it is carried onto

$V_i \cap H^1(K_i;\mathbb{Z})$ and this must be a full lattice subgroup since ψ_i surjects onto V_i. Recall that the rays issuing from the origin in $H^1(\widehat{W})$ that meet nonzero integer lattice points are called rational rays.

Lemma 5.1. *The union of rational rays in $H^1(\widehat{W})$ is everywhere dense.*

Proof. The open sets in $H^1(\widehat{W}) = \varprojlim V_i$ are unions of sets of the form

$$U = \varprojlim V_i \cap (U_0 \times U_1 \times \cdots U_n \times V_{n+1} \times V_{n+2} \times \cdots),$$

where $U_i \subseteq V_i$ is open, $0 \le i \le n$. If this set is nonempty, we must prove that it meets a rational ray. Let $\theta_i : V_n \to V_i$ be the natural surjection, $0 \le i < n$. Then the set

$$Y = U_n \cap \theta_{n-1}^{-1}(U_{n-1}) \cap \cdots \cap \theta_0^{-1}(U_0)$$

is open and the assumption that $U \ne \emptyset$ implies that $Y \ne \emptyset$. Since V_n is finite dimensional, the rational rays do have dense union there, hence we select such a ray ρ that meets Y. There is a rational ray ρ' in $H^1(\widehat{W})$ which is mapped onto ρ by ψ_n. Viewing ρ' in $\varprojlim V_i$, one sees that it meets U. $\square$

If $\mathcal{F}$ is only of class C^{0+}, Handel-Miller does not apply directly to the foliations of $\widehat{W}$ that are fibrations extending $\mathcal{F}|(M \smallsetminus W)$. Indeed, such a foliation may be nontrivial outside every K_i, hence the monodromy will not be endperiodic. It will be necessary to work in the progressively expanding K_i's, in each of which the endperiodic theory works fine, in order to produce the foliation cones in $H^1(\widehat{W}) = \varprojlim H^1(K_i)$. The reader will note that the lack of smoothness is only for the foliations in M, the foliation in each $\widehat{W}$ being itself of class $C^{\infty,\iota}$.

Let $\zeta \in H_1(\widehat{W})$. This class is represented by a singular cycle which necessarily lives in K_k, for large enough $k \ge 0$. Then $\zeta : H^1(K_k) \to \mathbb{R}$ is a continuous linear functional. We can define $\zeta : H^1(\widehat{W}) \to \mathbb{R}$ by the composition

$$H^1(\widehat{W}) \xrightarrow{\psi_k} H^1(K_k) \xrightarrow{\zeta} \mathbb{R},$$

remarking that this is independent of the choice of large enough k and defines a continuous linear functional.

We consider $\mathcal{F}|\widehat{W}$. While this foliation may never trivialize outside any K_i, $\mathcal{F}|K_i$ is a depth one foliation, hence is smooth and our previous discussion can be applied to this foliation. Note that, since we are working in the compact manifold K_i rather than $K_i^{\varnothing}$, all differential forms are compactly supported. We can assume that the arms are the connected components of $\widehat{W} \smallsetminus K_0^{\varnothing}$, hence that $\mathcal{F}$ is transverse to the interval fibers in these arms. We can choose the monodromy h so that, in the arms, it is given by the flow along these interval fibers and is Handel-Miller for $\mathcal{F}|K_0$, hence also for $\mathcal{F}|K_i$, $i \ge 0$. (One should note that a leaf of $\mathcal{F}$ may intersect K_i in a finite family of leaves of $\mathcal{F}|K_i$, but this really doesn't matter.) Thus, the core lamination $\mathcal{X}_h$ of $\mathcal{L}_h|K_i$ lives in K_0 and is independent of i. Exactly

as before, this lamination gives a set $[\gamma_1], [\gamma_2], \dots, [\gamma_r] \in H_1(\widehat{W})$, hence the linear inequalities $[\gamma_i] \geq 0$ define a polyhedral cone in $H^1(\widehat{W})$, the foliation cone we have designated by $\mathfrak{C}_h$ and now designate by $\mathfrak{C}_{\mathcal{F}}$.

Lemma 5.2. *The foliation cone $\mathfrak{C}_{\mathcal{F}}$ is polyhedral with finitely many faces.*

We have seen that $\operatorname{int}\mathfrak{C}_{\mathcal{F}}$ consists of classes of foliated forms. Any such form in the integer lattice restricts to an element of the integer lattice in each $H^1(K_i)$, hence has as period group a subgroup of $\mathbb{Z}$. Thus the rational rays in $\operatorname{int}\mathfrak{C}_{\mathcal{F}}$ correspond to fibrations of W that appropriately extend $\mathcal{F}|(M \smallsetminus W)$.

Proposition 5.3. *A proper foliated ray $\langle \mathcal{G} \rangle$ lies in $\operatorname{int}\mathfrak{C}_{\mathcal{F}}$ if and only if $\mathfrak{C}_{\mathcal{G}} = \mathfrak{C}_{\mathcal{F}}$.*

Proof. Suppose that $\langle \mathcal{G} \rangle$ ß $\operatorname{int}\mathfrak{C}_{\mathcal{F}}$. By Theorem 3.14, we choose $\mathcal{G}$ transverse to $\mathcal{L}_h$. The proof of Proposition 4.10 carries through for $\langle \mathcal{G}|K_i \rangle$ and $\mathfrak{C}^{\kappa}_{\mathcal{F}|K_i}$, $0 \leq i < \infty$, showing that $\mathcal{L}_h$ induces Handel-Miller monodromy for $\mathcal{G}|K_i$ also, $0 \leq i < \infty$. In particular, the same family $\gamma_1, \dots, \gamma_r$ defines the Handel-Miller cone for both $\mathcal{F}|K_i$ and $\mathcal{G}|K_i$, $0 \leq i < \infty$, hence $\mathfrak{C}_{\mathcal{F}} = \mathfrak{C}_{\mathcal{G}}$. Again, the converse is trivial. □

Corollary 5.4. *No proper foliated ray is contained in $\partial\mathfrak{C}_{\mathcal{F}}$.*

Indeed, this follows from Proposition 5.3 exactly as Corollary 4.11 follows from Proposition 4.10, using the denseness of the rational rays (Lemma 5.1).

Proposition 5.5. *The foliation cone $\mathfrak{C}_{\mathcal{F}}$ is maximal.*

Proof. As before, this will follow From Corollary 5.4 if the union of rays through the integer lattice that lie in the face F_i are dense in F_i, $1 \leq i \leq r$. This is proven analogously to Corollary 5.1. Simply add the constraint $[\gamma_i] = 0$. □

Proposition 5.6. *The rational rays in $\operatorname{int}\mathfrak{C}_{\mathcal{F}}$ each determine the corresponding foliation uniquely up to a C^0 isotopy in $\widehat{W}$ which is smooth in W.*

Proof. As before, this follows from the main result of [9]. That theorem must now be applied not only in K_0, but in each component of $K_i \smallsetminus K^{\varnothing}_{i-1}$, $i \geq 1$. For this, let $\mathcal{F}$ and $\mathcal{F}'$ correspond to the same proper foliated ray and choose $\mathcal{F}$ transverse to $\mathcal{L}_h$. Note that each component of ∂K_i, the interface between K_i and $\widehat{W} \smallsetminus K^{\varnothing}_i$, is either an $\mathcal{L}_h$-saturated rectangle or an $\mathcal{L}_h$-saturated annulus, hence transverse to $\mathcal{F}$. An application of the version of the Roussarie-Thurston theorem stated in [9, Theorem 4.6] gives an ambient isotopy of ∂K_i in K_i, fixing the boundary of each component pointwise, to a position transverse to $\mathcal{F}'$. The image of this isotopy may be assumed, via a small perturbation, to be disjoint from ∂K_{i-1}. Reversing the isotopy deforms $\mathcal{F}'$ to a foliation transverse to $\partial_{\pitchfork} K_{i-1}$, $i \geq 1$. Now the main result of [9] can be applied in K_0 and in each component of $K_i \smallsetminus K^{\varnothing}_{i-1}$, $i \geq 1$. □

The finiteness of the cones follows as before from the finiteness of the cones for K_0. The case in which W is a foliated product again gives that $H^1(\widehat{W})$ is itself the unique foliation cone. The proof of Theorem 1.4 for the rational rays is complete.

Remark. Recall that the substructure of $\mathcal{F}$ is the compact lamination $S = M \smallsetminus \mathcal{O}$. Refoliating in the product components of $\mathcal{O}$ is well understood, as noted above. In the finitely many non-product components, we have classified cohomologically the possible depth k refoliations. Thus the classification of finite depth foliations sharing the same substructure has been reduced to an essentially finite procedure in both the $C^{\infty,\iota}$ and C^{0+} categories.

6. Simple disk decompositions and foliation cones

An original goal of this work was to quantify the depth one foliations $\mathcal{F}$ of sutured manifolds M constructed by Gabai's process of disk decomposition [19]. In many cases, the foliation cones can be read off of the disks in a *simple* disk decomposition.

Definition 6.1. If the disks $D_1, \dots, D_n$ of the decomposition all live in M at the beginning, the disk decomposition of M is said to be *simple*

In other words, the decompositions can be done simultaneously rather than sequentially. Obviously, for this to be true, M must be a sutured handlebody and $\{[D_1], \dots, [D_n]\}$ß$H_2(M, \partial M)$ is a basis. If $\{[\alpha_1], \dots, [\alpha_n]\}$ß$H^1(M)$ is the Poincaré dual basis, $\langle [\alpha_1], \dots, [\alpha_n] \rangle$ will denote the convex polyhedral cone spanned by these vectors.

Proposition 6.2. *Let $\mathcal{F}$ be a depth one foliation constructed via the simple disk decomposition as above. Then every ray in* $\operatorname{int} \langle [\alpha_1], \dots, [\alpha_n] \rangle$ *is a foliated ray and $\langle \mathcal{F} \rangle$ is one of these.*

Proof. By Gabai's construction, each disk D_i is transverse to $\mathcal{F}$ except at a single positive (multi-pronged) saddle tangency. By the fact that the decomposition is simple, one sees that there are positive closed transversals σ_i to $\mathcal{F}$ meeting D_i precisely at the saddle tangency with $\int_{\sigma_i} \alpha_j = \delta_{ij}$, $1 \le i, j \le n$. Here, α_j is the usual Poincaré dual form to D_j, supported in a small normal neighborhood of D_j, $1 \le j \le n$. Since the intersection number of σ_i with a depth one leaf of $\mathcal{F}$ is strictly positive, it follows that the proper foliated ray $\langle \mathcal{F} \rangle$ lies in $\operatorname{int} \langle [\alpha_1], \dots, [\alpha_n] \rangle$. We wish to show that every ray in the interior of this cone through the integer lattice is a proper foliated ray.

Let ω be a foliated form defining $\mathcal{F}$. Then, since the saddle tangencies are positive, one constructs a smooth, nonvanishing vector field v on M such that

$$(t\omega + (1-t)\alpha_i)(v) > 0$$

on M, $0 < t \le 1$, $1 \le i \le n$. This can be done so that the 1-dimensional foliation $\mathcal{L}$ tangent to v is transverse both to $\mathcal{F}$ and D_i and, even though the

disks meet the annular components of $\partial_{\pitchfork} M$, one can arrange that $\mathcal{L}$ fibers these components over S^1. If $\mathcal{X}$ is the core lamination of $\mathcal{L}$, it follows that $t[\omega] + (1-t)[\alpha_i]$ takes strictly positive values on all nontrivial asymptotic cycles of $\mathcal{X}$, $0 < t \le 1$, $1 \le i \le n$. If f is the monodromy induced by $\mathcal{L}$ on a leaf of $\mathcal{F}$, we see that $t[\omega] + (1-t)[\alpha_i] \in \operatorname{int} \mathfrak{C}_f$, $0 < t \le 1$, $1 \le i \le n$. But these classes fill up the interior of $\langle [\alpha_1], \dots, [\alpha_n] \rangle$, proving that this is a subcone of $\mathfrak{C}_f$. □

In many examples, this proposition makes it possible to completely determine the foliation cones (cf. [11, Section 7] and [5, Section 5]). This involves relating the disks in the simple decomposition to the Markov process induced by the Handel-Miller monodromy.

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